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ON THE SPECTRUM OF RINGS OF FUNCTIONS

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Abstract. For D a domain and $E \subseteq D$, we investigate the prime spectrum of rings of functions from E to D, that is, of rings contained in $\prod_{e\in E} D$ and containing D. Among other things, we characterize, when M is a maximal ideal of finite index in D, those prime ideals lying above M which contain the kernel of the canonical map to $\prod_{e\in E}(D/M)$ as being precisely the prime ideals corresponding to ultrafilters on E. We give a sufficient condition for when all primes above M are of this form and thus establish a correspondence to the prime spectra of ultraproducts of residue class rings of D. As a corollary, we obtain a description using ultrafilters, differing from Chabert's original one which uses elements of the M-adic completion, of the prime ideals in the ring of integer-valued polynomials $\mathrm{Int}(D)$ lying above a maximal ideal of finite index.

1. Introduction

Let D be an integral domain, $E \subseteq D$, and $\mathcal{R}$ a subring of $\prod_{e\in E} D$, containing D. The elements of $\mathcal{R}$ can be interpreted as functions from E to D and, consequently, we call $\mathcal{R}$ a ring of functions from E to D. We will investigate the prime spectra of such rings of functions. We obtain, for quite general $\mathcal{R}$, a partial description of the prime spectrum, cf. Theorems 3.7 and 5.3, and in special cases a complete characterization, cf. Corollary 6.5.

Our motivation is the spectrum of a ring of integer-valued polynomials: For D an integral domain with quotient field K, let $\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}$ be the ring of integer-valued polynomials on D. More generally, when K is understood, we let $\mathrm{Int}(A, B) = \{f \in K[x] \mid f(A) \subseteq B\}$ for $A, B \subseteq K$.

If D is a Noetherian one-dimensional domain, a celebrated theorem of Chabert [1, Ch. V] states that every prime ideal of $\mathrm{Int}(D)$ lying over a maximal ideal M of finite index in D is maximal and of the form

$$M_\alpha = \{f \in \mathrm{Int}(D) \mid f(\alpha) \in \hat{M}\},$$

where α is an element of the M-adic completion $\hat{D}_M$ of D and $\hat{M}$ the maximal ideal of $\hat{D}_M$.

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In fact, Chabert showed two separate statements independently – both under the assumption that D is Noetherian and one-dimensional and M a maximal ideal of finite index of D:

(1) Every maximal ideal of $\mathrm{Int}(D)$ containing $\mathrm{Int}(D, M)$ is of the form M_α for some $\alpha \in \hat{D}_M$.
(2) Every maximal ideal of $\mathrm{Int}(D)$ lying over M contains $\mathrm{Int}(D, M)$.

For a simplified proof of Chabert's result, see [4], Lemma 4.4 and the remark following it.

We will show that a modified version of statement (1) holds in far greater generality, for rings of functions. The modification consists in replacing elements of the M-adic completion by ultrafilters.

Whether (2) holds or not for a particular D and a particular subring of D^E will have to be examined separately. It is, in some sense, a question of density of the subring in the product $\prod_{e\in E} D$.

We will work in the following setting:

Definition 1.1. *Let D be a commutative ring and $E \subseteq D$. Let $\mathcal{R}$ be a commutative ring and $\varphi\colon \mathcal{R} \to \prod_{e\in E} D$ a monomorphism of rings. φ allows us to interpret the elements of $\mathcal{R}$ as functions from E to D.*

If all constant functions are contained in $\varphi(\mathcal{R})$, we call the pair $(\mathcal{R}, \varphi)$ a ring of functions from E to D. We use $\mathcal{R} = \mathcal{R}(E, D)$ (where φ is understood) to denote a ring of functions from E to D.

Remark 1.2. *For our considerations it is vital that $\mathcal{R} = \mathcal{R}(E, D)$ contain all constant functions, because we will make extensive use of the following fact: when $\mathcal{I}$ is an ideal of $\mathcal{R} = \mathcal{R}(E, D)$, $f \in \mathcal{I}$ and $g \in D[x]$ a polynomial with zero constant term, then $g(f) \in \mathcal{I}$, and similarly, if g is a polynomial in several variables over D with zero constant term, and an element of $\mathcal{I}$ is substituted for each variable in g, then, an element of $\mathcal{I}$ results.*

Let us note that considerable research has been done on the spectrum of a power of a ring $D^E = \prod_{d\in E} D$ or a product of rings $\prod_{e\in E} D_e$. Gilmer and Heinzer [5, Prop. 2.3] have determined the spectrum of an infinite product of local rings, and Levy, Loustaunau and Shapiro [8] that of an infinite power of $\mathbb{Z}$. Our focus here is not on the full product of rings, but on comparatively small subrings and the question of how much information about the spectrum of a ring can be obtained from its embedding in a power of a domain.

One ring can be embedded in different products: $\mathrm{Int}(D)$ can be seen as a ring of functions from K to K as well as a ring of functions from D to D. We will glean a lot more information about the spectrum of $\mathrm{Int}(D)$ from the second interpretation than from the first.

2. Prime ideals corresponding to ultrafilters

Let $\mathcal{R} = \mathcal{R}(E, D)$ be a ring of functions from E to D as in Definition 1.1. We will now make precise the concept of ideals corresponding to ultrafilters, and the connection to ultraproducts $\prod_{e\in E}^{\mathcal{U}}(D/M)$, where M is a maximal ideal of D, and $\mathcal{U}$ an ultrafilter on E. First a quick review of filters, ultrafilters and ultraproducts:

Definition 2.1. *Let S be a set. A non-empty collection $\mathcal{F}$ of subsets of S is called a filter on S if*

(1) $\emptyset \notin \mathcal{F}$.
(2) $A, B \in \mathcal{F}$ *implies* $A \cap B \in \mathcal{F}$.
(3) $A \subseteq C \subseteq S$ *with* $A \in \mathcal{F}$ *implies* $C \in \mathcal{F}$.

A filter $\mathcal{F}$ on S is called an ultrafilter on S if, for every $C \subseteq S$, either $C \in \mathcal{F}$ or $S \setminus C \in \mathcal{F}$.

Let S be a fixed set and $\mathcal{P}(S)$ its power-set. For $C \in \mathcal{P}(S)$, a *superset* of C is a set $D \in \mathcal{P}(S)$ with $C \subseteq D \subseteq S$. A collection $\mathcal{C}$ of subsets of S is said to have the *finite intersection property* if the intersection of any finitely many members of $\mathcal{C}$ is non-empty.

Remark 2.2. *Clearly, a necessary and sufficient condition for $\mathcal{C} \subseteq \mathcal{P}(S)$ to be contained in a filter on S is that $\mathcal{C}$ satisfies the finite intersection property. If the finite intersection property is satisfied, then the supersets of finite intersections of members of $\mathcal{C}$ form a filter.*

Although, strictly speaking, we do not need ultraproducts to prove our results, we will nevertheless introduce them, because they provide context, in particular to Lemma 2.6, and to sections 3 and 5.

Definition 2.3. *Let S be an index set and $\mathcal{U}$ an ultrafilter on S. Suppose we are given, for each $s \in S$, a ring R_s. Then the ultraproduct of rings $\prod_{s\in S}^{\mathcal{U}} R_s$ is defined as the direct product $\prod_{s\in S} R_s$ modulo the congruence relation*

$$(r_s)_{s\in S} \sim (t_s)_{s\in S} \quad \Longleftrightarrow \quad \{s \in S \mid r_s = t_s\} \in \mathcal{U}.$$

Ultraproducts of other algebraic structures are defined analogously. The usefulness of ultraproducts is captured by the Theorem of Łoś (cf. [6, Chpt. 3.2] or [7, Prop 1.6.14]) which states that an ultraproduct $\prod_{s\in S}^{\mathcal{U}} R_s$ satisfies a first-order formula if and only if the set of indices s for which R_s satisfies the formula is in $\mathcal{U}$. Here first-order formula means a formula in the first-order language whose only non-logical symbols (apart from the equality sign) are symbols for the algebraic operations; for instance, $+$ and $\cdot$ in the case of an ultraproduct of rings.

Definition 2.4. *Let D be a domain, $E \subseteq D$, $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions, I an ideal of D and $\mathcal{F}$ a filter on E.*

For $f \in \mathcal{R}(E, D)$, we let $f^{-1}(I) = \{e \in E \mid f(e) \in I\}$ and define

$$I_{\mathcal{F}} = \{f \in \mathcal{R}(E, D) \mid f^{-1}(I) \in \mathcal{F}\}$$

Remark 2.5. *Let everything as in Definition 2.4, I, J ideals of D and $\mathcal{F}, \mathcal{G}$ filters on E. Some easy consequences of Definition 2.4 are:*

1. *If $I \neq D$ then $I_{\mathcal{F}} \neq \mathcal{R}$.*
2. *$I_{\mathcal{F}}$ is an ideal of $\mathcal{R}$ containing $\mathcal{R}(E, I) = \{f \in \mathcal{R} \mid f(E) \subseteq I\}$.*
3. *$I \subseteq J \Longrightarrow I_{\mathcal{F}} \subseteq J_{\mathcal{F}}$*
4. *$\mathcal{F} \subseteq \mathcal{G} \Longrightarrow I_{\mathcal{F}} \subseteq I_{\mathcal{G}}$*

Lemma 2.6. *Let D be a domain, $E \subseteq D$, and $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions from E to D.*

Then for every prime ideal P of D and every ultrafilter $\mathcal{U}$ on E, $P_{\mathcal{U}}$ is a prime ideal of $\mathcal{R}$.

Proof. Easy direct verification: let $fg \in P_{\mathcal{U}}$; because P is a prime ideal of D, the inverse image of P under $f \cdot g$ is the union of $f^{-1}(P)$ and $g^{-1}(P)$. If the union of two sets is in an ultrafilter, then one of them must be in the ultrafilter. Therefore, $f \in P_{\mathcal{U}}$ or $g \in P_{\mathcal{U}}$. Also, $P_{\mathcal{U}}$ cannot be all of $\mathcal{R}$ because it doesn't contain the constant function 1. □

One way of looking at $P_{\mathcal{U}}$ is by considering the following commuting diagram of ring-homomorphisms, where π and π_1 mean applying the canonical projection in each factor of the product, and σ and σ_1 mean factoring through the defining congruence relation of an ultraproduct.

$$\begin{array}{ccccc} \mathcal{R} & \xrightarrow{\varphi} & \prod_{e\in E} D & \xrightarrow{\sigma_1} & \prod^{\mathcal{U}}_{e\in E} D \\ & & \downarrow \pi & & \downarrow \pi_1 \\ & & \prod_{e\in E}(D/P) & \xrightarrow{\sigma} & \prod^{\mathcal{U}}_{e\in E}(D/P) \end{array}$$

$P_{\mathcal{U}}$ is the kernel of the following composition of ring homomorphisms:

$$\varphi\colon \mathcal{R} \to \prod_{e\in E} D$$

followed by the canonical projection

$$\pi\colon \prod_{e\in E} D \to \prod_{e\in E}(D/P)$$

and the canonical projection

$$\sigma\colon \prod_{e\in E}(D/P) \to \prod^{\mathcal{U}}_{e\in E}(D/P)$$

Since D/P is an integral domain, any ultraproduct of copies of D/P is also an integral domain, by the Theorem of Łoś. Therefore (0) is a prime ideal of $\prod^{\mathcal{U}}_{e\in E}(D/P)$ and hence $P_{\mathcal{U}}$ a prime ideal of $\mathcal{R}$. We also see that $P_{\mathcal{U}}$ is the inverse image of a prime ideal of $\prod_{e\in E} D$ under φ, and further, of a prime ideal of the ultraproduct $\prod_{e\in E}^{\mathcal{U}} D$ under $\sigma_1 \circ \varphi$.

3. The set of zero-loci mod M of an ideal of the ring of functions

As before, D is a domain with quotient field K, $E \subseteq D$ and $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions from E to D as in Def. 1.1. Especially, recall from Def. 1.1 that $\mathcal{R}$ is assumed to contain all constant functions.

Definition 3.1. *For $M \subseteq D$ and $f \in \mathcal{R} = \mathcal{R}(E, D)$, let*

$$f^{-1}(M) = \{e \in E \mid f(e) \in M\}.$$

For an ideal M of D and an ideal $\mathcal{I}$ of $\mathcal{R}$, let

$$\mathcal{Z}_M(\mathcal{I}) = \{f^{-1}(M) \mid f \in \mathcal{I}\}$$

Recall from Def. 2.4 that for a filter $\mathcal{F}$ on E,

$$M_{\mathcal{F}} = \{f \in \mathcal{R}(E, D) \mid f^{-1}(M) \in \mathcal{F}\}$$

Remark 3.2. *Note that the above definition implies*

(1) $\mathcal{I} \subseteq \mathcal{J} \Longrightarrow \mathcal{Z}_M(\mathcal{I}) \subseteq \mathcal{Z}_M(\mathcal{J})$
(2) $\mathcal{I} \subseteq M_{\mathcal{F}} \Longleftrightarrow \mathcal{Z}_M(\mathcal{I}) \subseteq \mathcal{F}$

Lemma 3.3. *Let M be an ideal of D and $\mathcal{I}$ an ideal of $\mathcal{R}$. The following are equivalent:*

(a) *There exists a filter $\mathcal{F}$ on E such that $\mathcal{I} \subseteq M_{\mathcal{F}}$.*
(b) *$\mathcal{Z}_M(\mathcal{I})$ satisfies the finite intersection property.*

Proof. If $\mathcal{I} \subseteq M_{\mathcal{F}}$, then $\mathcal{Z}_M(\mathcal{I})$ is contained in $\mathcal{F}$ and hence satisfies the finite intersection property. Conversely, if $\mathcal{Z}_M(\mathcal{I})$ satisfies the finite intersection property then, by Remark 2.2, the supersets of finite intersections of sets in $\mathcal{Z}_M(\mathcal{I})$ form a filter $\mathcal{F}$ on E for which $\mathcal{Z}_M(\mathcal{I}) \subseteq \mathcal{F}$ and hence $\mathcal{I} \subseteq M_{\mathcal{F}}$. □

In the case where $\mathcal{R}(E, D) = \prod_{e\in E} D$ is the ring of all functions from E to D, much more can be said; see the papers by Gilmer and Heinzer [5, Prop. 2.3] (concerning local rings) and Levy, Loustaunau and Shapiro [8] (concerning $D = \mathbb{Z}$).

For a field K that is not algebraically closed, we will need, for an arbitrary $n \geq 2$, an n-ary form that has no zero but the trivial one. For this purpose, recall how to define a norm form: if $L : K$ is an n-dimensional field extension, multiplication by any $w \in L$ is a K-endomorphism ψ_w of L. For a fixed choice of a K-basis of L, map every $w \in L$ to the determinant of the matrix of ψ_w with respect to the chosen basis. This mapping, regarded as a function of the coordinates of w with respect to the chosen basis, is easily seen to be an n-ary form that has no zero but the trivial one.

Lemma 3.4. *Let M be a maximal ideal of D such that D/M is not algebraically closed. Then for every ideal $\mathcal{I}$ of $\mathcal{R} = \mathcal{R}(E, D)$, $\mathcal{Z}_M(\mathcal{I})$ is closed under finite intersections.*

Proof. Given $f, g \in \mathcal{I}$, we show that there exists $h \in \mathcal{I}$ with

$$h^{-1}(M) = f^{-1}(M) \cap g^{-1}(M).$$

Consider any finite-dimensional non-trivial field extension of D/M, and let n be the degree of the extension. The norm form of this field extension is a homogeneous polynomial in $n \geq 2$ indeterminates whose only zero in $(D/M)^n$ is the trivial one. By identifying $n-1$ variables, we get a binary form $\bar{s} \in (D/M)[x, y]$ with no zero in $(D/M)^2$ other than $(0, 0)$. Let $s \in D[x, y]$ be a binary form that reduces to $\bar{s}$ when the coefficients are taken mod M.

Now, given f and g in $\mathcal{I}$, we set $h = s(f, g)$. By the fact that $\mathcal{R}$ contains all constant functions, h is in $\mathcal{I}$. Also, $h(e) \in M$ if and only if both $f(e) \in M$ and $g(e) \in M$, as desired. □

Lemma 3.5. *Let M be a maximal ideal of D and $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions such that every $f \in \mathcal{R}$ takes values in only finitely many residue classes mod M.*

Then for every ideal $\mathcal{I}$ of $\mathcal{R}$, $\mathcal{Z}_M(\mathcal{I})$ is closed under finite intersections.

Proof. Again, given $f, g \in \mathcal{I}$, we show that there exists $h \in \mathcal{I}$ with

$$h^{-1}(M) = f^{-1}(M) \cap g^{-1}(M).$$

Let $A, B \subseteq D/M$ be finite sets of residue classes of D mod M such that $f(E)$ is contained in the union of A and $g(E)$ in the union of B.

We can interpolate any function from $(D/M)^2$ to (D/M) at any finite set of arguments by a polynomial in $(D/M)[x, y]$. Pick $\bar{s} \in (D/M)[x, y]$ with $\bar{s}(0, 0) = 0$ and $\bar{s}(a, b) = 1$ for all $(a, b) \in (A \times B) \setminus \{(0, 0)\}$. Let $s \in D[x, y]$ be a polynomial with zero constant coefficient that reduces to $\bar{s}$ when the coefficients are taken mod M.

Now, given f and g in $\mathcal{I}$, we set $h = s(f, g)$. By the fact that $\mathcal{R}$ contains all constant functions, h is in $\mathcal{I}$. Also, $h(e) \in M$ if and only if both $f(e) \in M$ and $g(e) \in M$, as desired. □

Definition 3.6. *Let $\mathcal{R} = \mathcal{R}(E, D)$ be a ring of functions and M an ideal of D. We call $f \in \mathcal{R}$ an M-unit-valued function if $f(e) + M$ is a unit in D/M for every $e \in E$.*

Theorem 3.7. *Let M be a maximal ideal of D and $\mathcal{I}$ an ideal of $\mathcal{R} = \mathcal{R}(E, D)$. Assume that either D/M is not algebraically closed or that each function in $\mathcal{R}$ takes values in only finitely many residue classes mod M.*

1. *$\mathcal{I}$ is contained in an ideal of the form $M_{\mathcal{F}}$ for some filter $\mathcal{F}$ on E if and only if $\mathcal{I}$ contains no M-unit-valued function.*
2. *Every ideal $\mathcal{Q}$ of $\mathcal{R}$ that is maximal with respect to not containing any M-unit-valued function is of the form $M_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$ on E.*

(3) *In particular, every maximal ideal of* $\mathcal{R}$ *that does not contain any* M*-unit-valued function is of the form* $M_{\mathcal{U}}$ *for some ultrafilter* $\mathcal{U}$ *on* E.

Proof. Ad (1). If $\mathcal{I}$ is contained in an ideal of the form $M_{\mathcal{F}}$, $\mathcal{I}$ cannot contain any M-unit-valued function, because $\mathcal{F}$ doesn't contain the empty set.

Conversely, suppose that $\mathcal{I}$ does not contain any M-unit-valued function. Then $\emptyset \notin \mathcal{Z}_M(\mathcal{I})$. By Lemmata 3.4 and 3.5, $\mathcal{Z}_M(\mathcal{I})$ is closed under finite intersections. $\mathcal{Z}_M(\mathcal{I})$, therefore, satisfies the finite intersection property. By Remark 2.2, $\mathcal{Z}_M(\mathcal{I})$ is contained in a filter $\mathcal{F}$ on E. For this filter, $\mathcal{I} \subseteq M_{\mathcal{F}}$, by Remark 3.2.

Ad (2). Suppose $\mathcal{Q}$ is maximal with respect to not containing any M-unit-valued function. By (1), $\mathcal{Q} \subseteq M_{\mathcal{F}}$ for some filter $\mathcal{F}$. Refine $\mathcal{F}$ to an ultrafilter $\mathcal{U}$. Then, by Remark 2.5, $\mathcal{Q} \subseteq M_{\mathcal{F}} \subseteq M_{\mathcal{U}}$, and $M_{\mathcal{U}}$ doesn't contain any M-unit-valued function. Since $\mathcal{Q}$ is maximal with this property, $\mathcal{Q} = M_{\mathcal{U}}$.

(3) is a special case of (2). □

4. A dichotomy of maximal ideals

In what follows, D is always a domain with quotient field K, $E \subseteq D$ and $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions from E to D as in Def. 1.1. When the interpretation of $\mathcal{R}$ as a subring of $\prod_{e\in E} D$ is understood, then for $M \subseteq D$ we let

$$\mathcal{R}(E, M) = \{f \in \mathcal{R} \mid f(E) \subseteq M\}.$$

Proposition 4.1. *Let* M *be a maximal ideal of* D *and* $\mathcal{Q}$ *a maximal ideal of* $\mathcal{R} = \mathcal{R}(E, D)$. *Then exactly one of the following two statements holds:*

(1) $\mathcal{Q}$ *contains* $\mathcal{R}(E, M) = \{f \in \mathcal{R} \mid f(E) \subseteq M\}$

(2) $\mathcal{Q}$ *contains an element* f *with* $f(e) \equiv 1 \bmod M$ *for all* $e \in E$.

Proof. The two cases are mutually exclusive, because any ideal $\mathcal{Q}$ satisfying both statements must contain 1.

Now suppose $\mathcal{Q}$ does not contain $\mathcal{R}(E, M)$. Let $g \in \mathcal{R}(E, M) \setminus \mathcal{Q}$. By the maximality of $\mathcal{Q}$, $1 = h(x)g(x) + f(x)$ for some $h \in \mathcal{R}$ and $f \in \mathcal{Q}$. We see that $f(x) = 1 - h(x)g(x) \in \mathcal{Q}$ satisfies $f(e) \equiv 1 \bmod M$ for all $e \in E$. □

Recall that a function $f \in \mathcal{R}$ is called M-unit-valued if $f(e) + M$ is a unit in D/M for every $e \in E$.

Lemma 4.2. *Let* M *be an ideal of* D *and* $\mathcal{Q}$ *an ideal of* $\mathcal{R} = \mathcal{R}(E, D)$. *The following are equivalent:*

(A) $\mathcal{Q}$ *contains an element* f *with* $f(e) \equiv 1 \bmod M$ *for all* $e \in E$.

(B) $\mathcal{Q}$ *contains an* M*-unit-valued function that takes values in only finitely many residue classes mod* M.

Proof. To see that the a priori weaker statement implies the stronger, let $g \in \mathcal{Q}$ be an M-unit-valued function taking only finitely many different values mod M. Let $d_1, \ldots, d_k \in D$ be representatives of the finitely many residue classes mod M intersecting $g(E)$ non-trivially, and $u \in D$ an inverse mod M of $(-1)^{k+1} d_1 \cdot \ldots \cdot d_k$.

Then

$$h(x) = \prod_{i=1}^{k}(g(x) - d_i) - (-1)^k d_1 \cdot \ldots \cdot d_k$$

is in $\mathcal{Q}$ and $h(e) \equiv (-1)^{k+1} d_1 \cdot \ldots \cdot d_k \bmod M$ for all $e \in E$. Therefore $f(x) = uh(x) \in \mathcal{Q}$ satisfies $f(e) \equiv 1 \bmod M$ for all $e \in E$. □

Proposition 4.3. *Let M be a maximal ideal of D and $\mathcal{Q}$ a maximal ideal of $\mathcal{R} = \mathcal{R}(E, D)$. If each $f \in \mathcal{R}$ takes values in only finitely many residue classes mod M (in particular, if D/M happens to be finite) then exactly one of the following statements holds:*

(1) *$\mathcal{Q}$ contains $\mathcal{R}(E, M) = \{f \in \mathcal{R} \mid f(E) \subseteq M\}$*
(2) *$\mathcal{Q}$ contains an M-unit-valued function.*

Proof. This follows directly from Proposition 4.1 and Lemma 4.2. □

The Propositions in this section partition the maximal ideals of $\mathcal{R}$ lying over a maximal ideal M of D into two types: those containing $\mathcal{R}(E, M)$ (the kernel of the restriction to $\mathcal{R}$ of the canonical projection $\pi\colon \prod_{e\in E} D \longrightarrow \prod_{e\in E}(D/M)$), and the others.

In some cases, it is known that all maximal ideals of $\mathcal{R}$ lying over M contain $\mathcal{R}(E, M)$, notably if $\mathcal{R} = \mathrm{Int}(D)$ and M is finitely generated and of finite index in D [1, Ch. V], [4, Lemma 4.4]. We will find a sufficient condition for all maximal ideals of $\mathcal{R}$ lying over M to contain $\mathcal{R}(E, M)$ in Theorem 6.4.

We must not discount the possibility of a maximal ideal $\mathcal{Q}$ lying over M containing an M-unit-valued function, however. If D is an infinite domain, $D[x]$ is embedded in D^D by mapping every polynomial to the corresponding polynomial function. When D/M is not algebraically closed, then there are certainly maximal ideals of $D[x]$ lying over M that contain polynomials without a zero mod M.

5. Prime ideals containing $\mathcal{R}(E, M)$

We are now in a position to characterize the prime ideals of $\mathcal{R}$ containing $\mathcal{R}(E, D)$ as being precisely the ideals of the form $M_{\mathcal{U}}$ for ultrafilters $\mathcal{U}$ on E, under the following hypothesis: every $f \in \mathcal{R}$ takes values in only finitely many residue classes of M.

This hypothesis may seem only marginally weaker than the assumption that D/M is finite. Note however, that it is sometimes satisfied for infinite D/M under perfectly natural circumstances, for instance, when E intersects only finitely many residue classes of M^n for each $n \in \mathbb{N}$ (E precompact), and $\mathcal{R}$ consists of functions that are uniformly M-adically continuous.

As in the case of integer-valued polynomials, we can show that every prime ideal of $\mathcal{R}(E, D)$ containing $\mathcal{R}(E, M)$ is maximal under certain conditions, notably if D/M is finite. The proof for $\mathrm{Int}(D)$, when D/M is finite [1, Lemma V.1.9.], carries over practically without change. Note that Definition 1.1 ensures that every ring of functions $\mathcal{R}$ contains all constant functions – an essential requirement of the following proof.

Lemma 5.1. *Let M be a maximal ideal of D such that every function in $\mathcal{R} = \mathcal{R}(E, D)$ takes values in only finitely many residue classes mod M, and $\mathcal{Q}$ a prime ideal of $\mathcal{R}(E, D)$ containing $\mathcal{R}(E, M)$. Then $\mathcal{Q}$ is maximal and $\mathcal{R}/\mathcal{Q}$ is isomorphic to D/M.*

Proof. Let $\mathcal{Q}$ be a prime ideal of $\mathcal{R}(E, D)$ containing $\mathcal{R}(E, M)$, and A a system of representatives of D mod M. It suffices to show that A (viewed as a set of constant functions) is also a system of representatives of $\mathcal{R}$ mod $\mathcal{Q}$. Let $f \in \mathcal{R}(E, D)$ and $a_1, \ldots, a_r \in A$ the representatives of those residue classes of M that intersect $f(E)$ non-trivially. Then $\prod_{i=1}^{r}(f - a_i)$ is in $\mathcal{R}(E, M) \subseteq \mathcal{Q}$ and, $\mathcal{Q}$ being prime, one of the factors $(f - a_i)$ must be in $\mathcal{Q}$. This shows that f is congruent mod $\mathcal{Q}$ to one of the constant functions $a_1, \ldots, a_r$, and, in particular, to an element of A. Therefore, A is a system of representatives of $\mathcal{R}(E, D)$ mod $\mathcal{Q}$. □

Lemma 5.2. *Let $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions and M a maximal ideal of D such that every $f \in \mathcal{R}$ takes values in only finitely many residue classes of M. Let $\mathcal{I}$ be an ideal of $\mathcal{R}$.*

Then $\mathcal{I}$ is contained in an ideal of the form $M_{\mathcal{F}}$ for a filter $\mathcal{F}$ on E if and only if $\mathcal{R}(E, M) \subseteq \mathcal{I}$.

Proof. $\mathcal{R}(E, M) \subseteq \mathcal{I}$ is equivalent to $\mathcal{I}$ not containing an M-unit-valued function, by Proposition 4.3. The statement therefore follows from part (1) of Theorem 3.7. □

Theorem 5.3. *Let $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions, and M a maximal ideal of D. If every $f \in \mathcal{R}$ takes values in only finitely many residue classes of M (and, in particular, if D/M is finite), then the prime ideals of $\mathcal{R}$ containing $\mathcal{R}(E, M)$ are exactly the ideals of the form $M_{\mathcal{U}}$ with $\mathcal{U}$ an ultrafilter on E. Each of them is maximal and its residue field isomorphic to D/M.*

Proof. Let $\mathcal{Q}$ be a prime ideal of $\mathcal{R}$ containing $\mathcal{R}(E, M)$. By Lemma 5.1, $\mathcal{Q}$ is maximal and $\mathcal{R}/\mathcal{Q}$ is isomorphic to D/M. By Lemma 5.2, $\mathcal{Q} \subseteq M_{\mathcal{F}}$ for some filter $\mathcal{F}$ on E. $\mathcal{F}$ can be refined to an ultrafilter $\mathcal{U}$ on E, and then $\mathcal{Q} \subseteq M_{\mathcal{F}} \subseteq M_{\mathcal{U}} \neq \mathcal{R}$, by Remark 2.5. Since $\mathcal{Q}$ is maximal, $\mathcal{Q} = M_{\mathcal{U}}$ follows.

Conversely, every ideal of the form $M_{\mathcal{U}}$ for an ultrafilter $\mathcal{U}$ on E is prime, by Lemma 2.6, and contains $\mathcal{R}(E, M)$, by Remark 2.5. □

Note, in particular, that Theorems 3.7 and 5.3 apply to $\mathcal{R} = \mathrm{Int}(E, D)$. In this way, we see, when M is a maximal ideal of finite index in D, that prime ideals of $\mathrm{Int}(E, D)$ containing $\mathrm{Int}(D, M)$ are inverse images of prime ideals of D^E, and ultimately come from ultrapowers of (D/M), as in the discussion after Lemma 2.6.

6. Divisible rings of functions

Let $\mathcal{R} \subseteq D^E$ be a ring of functions and M a maximal ideal of D. We have seen that we can describe those maximal ideals of $\mathcal{R}$ lying over M that contain $\mathcal{R}(E, M)$. We would like to know under what conditions this holds for every maximal ideal of $\mathcal{R}$ lying over M.

In the case where M is a maximal ideal of finite index in a one-dimensional Noetherian domain D, Chabert showed that every maximal ideal of $\mathrm{Int}(D)$ lying over M contains $\mathrm{Int}(D, M)$, cf. [1, Prop. V.1.11] and [4, Lemma 3.3]. Once we know this, Theorem 5.3 is applicable. It can be used to give an alternative proof of the fact that every prime ideal of $\mathrm{Int}(D)$ lying over M is maximal and of the form $M_\alpha = \{f \in \mathrm{Int}(D) \mid f(\alpha) \in \hat{M}\}$ for an element α in the M-adic completion of D.

We will now generalize Chabert's argument from integer-valued polynomials to a class of rings of functions which we call divisible. Note that we do not have to restrict ourselves to Noetherian domains; we only require the individual maximal ideal for which we study the primes of $\mathcal{R}$ lying over it to be finitely generated. It is true that our questions only localize well when the domain is Noetherian, but we will pursue a different course, not relying on localization.

Definition 6.1. *Let R be a commutative ring and $E \subseteq R$. We call a ring of functions $\mathcal{R} \subseteq R^E$* ***divisible*** *if it has the following property: If $f \in \mathcal{R}$ is such that $f(E) \subseteq cR$ for some non-zero $c \in R$, then every function $g \in R^E$ satisfying $cg(x) = f(x)$ is also in $\mathcal{R}$.*

We call $\mathcal{R}$ ***weakly divisible*** *if for every $f \in \mathcal{R}$ and every non-zero $c \in R$ such that $f(E) \subseteq cR$, there exists a function $g \in \mathcal{R}$ with $cg(x) = f(x)$.*

If R is a domain, we note that $g(x)$ in the above definition is unique and that, therefore, for domains, weakly divisible is equivalent to divisible.

Example 6.2.
(1) *$\mathrm{Int}(E, D)$ is divisible. - This is our motivation.*
(2) *If D is a valuation domain with maximal ideal M then the ring of uniformly M-adically continuous functions from E to D ($E \subseteq D$ equipped with subspace topology of M-adic topology) is a divisible ring of functions.*

We now consider minimal prime ideals of non-zero principal ideals, that is, P containing some $p \neq 0$ such that there is no prime ideal strictly contained in P and containing p. If D is Noetherian, this condition reduces to "$\mathrm{ht}(P) = 1$". In non-Noetherian domains, we find examples with $\mathrm{ht}(P) > 1$, for instance, the maximal ideal of a finite-dimensional valuation domain.

Lemma 6.3. *Let R be a domain, P a finitely generated prime ideal that is a minimal prime of a non-zero principal ideal $(p) \subseteq P$. Then there exist $m \in \mathbb{N}$ and $s \in R \setminus P$ such that $sP^m \subseteq pR$.*

Proof. In the localization R_P, P_P is the radical of pR_P. Therefore, since P (and hence P_P) is finitely generated, there exists $m \in \mathbb{N}$ with ${P_P}^m \subseteq pR_P$ and in particular $P^m \subseteq pR_P$. The ideal P^m is also finitely generated, by $p_1, \ldots, p_k$, say. Let $a_i \in R_P$ with $p_i = pa_i$. By considering the fractions $a_i = r_i/s_i$ (with $r_i \in R$ and $s_i \in R \setminus P$), and setting $s = s_1 \cdot \ldots \cdot s_k$, we see that $sP^m \subseteq pR$ as desired. □

Theorem 6.4. *Let D be a domain and P a finitely generated prime ideal that is a minimal prime of a non-zero principal ideal. Let $\mathcal{R} \subseteq D^E$ be a divisible ring of functions from E to D. Then every prime ideal $\mathcal{Q}$ of $\mathcal{R}$ with $\mathcal{Q} \cap D = P$ contains $\mathcal{R}(E, P)$.*

Proof. Let $f \in \mathcal{R}(E, P)$. Let $p \in P$ non-zero and such that there is no prime ideal P_1 with $(p) \subseteq P_1 \subsetneq P$. By Lemma 6.3, there are $m \in \mathbb{N}$ and $s \in D \setminus P$ such that $sP^m \subseteq pD$. Then $sf^m \in \mathcal{R}(E, pD)$. Since $\mathcal{R}$ is divisible, $sf^m = pg$ for some $g \in \mathcal{R}(E, D)$. Therefore, $sf^m \in p\mathcal{R}(E, D) \subseteq \mathcal{Q}$. As $\mathcal{Q}$ is prime and $s \notin \mathcal{Q}$, we conclude that $f \in \mathcal{Q}$. □

Corollary 6.5. *Let D be a domain, M a finitely generated maximal ideal of height 1, and E a subset of D. Let $\mathcal{R} \subseteq D^E$ be a divisible ring of functions from E to D, such that each $f \in \mathcal{R}$ takes its values in only finitely many residue classes of M in D.*

Then the prime ideals of $\mathcal{R}$ lying over M are precisely the ideals of the form $M_{\mathcal{U}}$ for an ultrafilter $\mathcal{U}$ on E. Each $M_{\mathcal{U}}$ is a maximal ideal and its residue field isomorphic to D/M.

Proof. This follows from Theorem 6.4 via Theorem 5.3. □

To summarize, we can, using ultrafilters, describe certain prime ideals of a ring of functions $\mathcal{R} = \mathcal{R}(E, D)$ lying over a maximal ideal M pretty well: namely, those prime ideals that do not contain M-unit-valued functions (Theorem 3.7), or that contain $\mathcal{R}(E, M)$ (Theorem 5.3).

We have, so far, little information about when all prime ideals of $\mathcal{R}$ lying over M are of this form, apart from the sufficient condition in Theorem 6.4.

If we restrict our attention to rings of functions $\mathcal{R}$ with $D[x] \subseteq \mathcal{R}(E, D) \subseteq D^E$, it would be interesting to find a precise criterion, perhaps involving topological density, for this property.

Note that in the "nicest" case, that of $\mathrm{Int}(D)$, where D is a Dedekind ring with finite residue fields, not only is $\mathrm{Int}(D, M)$ contained in every prime ideal of $\mathrm{Int}(D)$ lying over a maximal ideal M of D, but also $\mathrm{Int}(D)$ is dense in D^D with product topology of discrete topology on D [2, 3].

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