

SPLIT ABSOLUTELY IRREDUCIBLE INTEGER-VALUED POLYNOMIALS OVER DISCRETE VALUATION DOMAINS

SOPHIE FRISCH, SARAH NAKATO, AND ROSWITHA RISSNER

ABSTRACT. Regarding non-unique factorization of integer-valued polynomials over a discrete valuation domain with finite residue field, it is known that there exist absolutely irreducible elements, that is, irreducible elements all of whose powers factor uniquely, and non-absolutely irreducible elements.

We completely and constructively characterize the absolutely irreducible elements among split integer-valued polynomials. They correspond bijectively to finite sets with a certain property regarding M -adic topology. For each such “balanced” set of roots, there exists a unique vector of multiplicities and a unique constant so that the corresponding product of monic linear factors times the constant is an absolutely irreducible integer-valued polynomial. This also yields sufficient criteria for integer-valued polynomials over Dedekind domains to be absolutely irreducible.

1. INTRODUCTION

In rings with non-unique factorization into irreducibles there usually exist irreducible elements some of whose powers have factorizations into irreducibles other than the obvious one. They are called non-absolutely irreducible (cf. Definition 2.10). To understand patterns of non-unique factorizations it is important to identify the non-absolutely irreducible elements. In rings of integer-valued polynomials

$$\text{Int}(R) = \{f \in K[x] \mid f(R) \subseteq R\},$$

where R is a domain with quotient field K , examples of both absolutely irreducible and non-absolutely irreducible elements have been given, for instance by the second author of this paper [16]. Note that absolutely irreducible elements are also called strong atoms, for instance, by Chapman and Krause [7], or completely irreducible, by Kaczorowski [13]. Some of Nakato’s examples concern polynomials that split over K .

In this paper, we completely characterize split absolutely irreducible integer-valued polynomials over a discrete valuation domain (R, M) with finite residue field. It suffices to consider polynomials whose roots are elements of R , as the only split absolutely irreducible integer-valued polynomials with roots in $K \setminus R$ are linear polynomials and hence easily understood. We, therefore, consider polynomials of the form

$$F = c^{-1}f \text{ with } f = \prod_{s \in S} (x - s)^{m_s} \quad (1)$$

where $S \subseteq R$ is a finite set and $c \in R$. We show in Theorem 2 that F is absolutely irreducible in $\text{Int}(R)$ if and only if c is a generator of the fixed divisor of f , S is a balanced set, and f the equalizing polynomial of S . Balanced refers to the

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distribution of S among residue classes of the powers of M (Definition 4.2) and the equalizing polynomial results from a particular choice of multiplicities $(m_s)_{s \in S}$ (Definition 6.8), and the fixed divisor is the ideal generated by the image of f (Definition 2.2).

So far the study of non-unique factorization has focused on Krull monoids, which are characterized by having a so-called “divisor theory”. Among integral domains, Krull domains are exactly the ones whose multiplicative monoids $D \setminus \{0\}$ are Krull monoids, cf. [12].

The ring $\text{Int}(R)$ of integer-valued polynomials over a discrete valuation domain (with finite residue field) is known to be Prüfer, but not Krull, cf. [6, 14]. What is Krull, however, is the monadic submonoid generated by a single polynomial f , that is,

$$\llbracket f \rrbracket = \{g \in \text{Int}(D) \mid g \text{ divides } f^n \text{ for some } n \in \mathbb{N}\}.$$

This submonoid contains all the information about all factorizations of the powers of a single polynomial f . (Reinhart [19] proved that the monadic submonoid of $\text{Int}(R)$ is Krull for factorial domains R and Frisch [9] extended this result to Krull domains R .)

Earlier work on factorization-theoretic properties in rings of integer-valued polynomials can be found in the work of Cahen and Chabert [3], Anderson, Cahen, Chapman and Smith [1] as well as Chapman and McClain [8]. For a thorough introduction into the theory of integer-valued polynomials we refer to the textbook of Cahen and Chabert [4] and their more recent survey [5].

Returning to absolute irreducibility, it is immediately seen that f is absolutely irreducible if and only if the monadic submonoid $\llbracket f \rrbracket$ of f is factorial, which is again equivalent to the divisor class group of $\llbracket f \rrbracket$ being trivial. Our results, therefore, add to the insight into the monadic submonoids of rings of integer-valued polynomials.

Moreover, absolutely irreducible elements play an important role when it comes to the construction of elements with a certain factorization behaviour. For rings of integer-valued polynomials, only little is known so far about absolutely irreducible polynomials.

Frisch and Nakato [10] give a graph-theoretic criterion for an integer-valued polynomial with square-free denominator over a principal ideal domain to be absolutely irreducible. For polynomials with squares appearing in the denominator, the graph-theoretic condition is shown to be sufficient, but not necessary. One consequence of their result is that the binomial polynomial $\binom{x}{p}$ is absolutely irreducible in $\text{Int}(\mathbb{Z})$ for a prime number p . The latter has been shown before by McClain [15].

In the general case $\binom{x}{n}$ their graph-theoretic argument is not applicable due to the nature of the denominator. Recently, Rissner and Windisch [20] verified that the binomial polynomials $\binom{x}{n}$ are indeed absolutely irreducible in $\text{Int}(\mathbb{Z})$ for any $n \in \mathbb{N}$. For the special case where $n = p^k$ is a prime power, Corollary 8.9 below serves as an alternative proof.

Here is a brief outline of our strategy for characterizing split absolutely irreducible polynomials. We set aside those that have a root in $K \setminus R$. (We will later show that they are all linear, see Corollary 7.2). From then on we only consider polynomials that split over R . We define balanced sets (Definition 4.2) and establish a host of somewhat technical facts about them (Sections 4, 5). These facts allow us to find for each balanced set S a vector of multiplicities $(m_s)_{s \in S}$ (Proposition 6.1) such that $\prod_{s \in S} (x - s)^{m_s}$ multiplied by an appropriate constant is absolutely irreducible (Theorem 1). We show that this vector of multiplicities is unique because a certain type of matrix is non-singular (Proposition 6.5). Finally, in Theorem 2, we prove that all split absolutely irreducible polynomials with roots in R are of this kind by showing that the set of roots always is a balanced set.

2. PRELIMINARIES

Convention 2.1. Throughout this paper, unless explicitly stated otherwise, D is always a domain with quotient field K and (R, M) always denotes a discrete valuation domain with quotient field K and finite residue field.

2.1. Integer-valued polynomials. Note that we only introduce the basic notions concerning integer-valued polynomials required in this work. A detailed treatment of the theory of integer-valued polynomials can be found in the textbook of Cahen and Chabert [4] and their recent survey [5].

Definition 2.2. Let D be a domain with quotient field K . The *ring of integer-valued polynomials on D* is defined as

$$\text{Int}(D) = \{F \in K[x] \mid F(D) \subseteq D\}.$$

For $F \in \text{Int}(D)$, the *fixed divisor* of F is the ideal

$$d(F) = (F(a) \mid a \in D)$$

of D generated by the elements $F(a)$ with $a \in D$.

The polynomial F is said to be *image-primitive* if $d(F) = D$.

Remark 2.3. Let $f \in D[x]$ and $b \in D$. Then

$$\frac{f}{b} \in \text{Int}(D) \iff d(f) \subseteq bD$$

Remark 2.4. Let D be a domain and $g \in D[x]$. If P is a prime ideal of D such that $d(g) \subseteq P$, then $g \in P[x]$ or $\|P\| = |D/P| \leq \deg(g)$.

Remark 2.5. Let D be a principal ideal domain with quotient field K .

- (1) All fixed divisors are principal ideals. Below, the notation d_f stands for an arbitrary but fixed generator of the fixed divisor of f .
- (2) There is for every $f \in D[x]$ an image-primitive polynomial $F \in \text{Int}(D)$ (unique up to multiplication by units of D) associated to f in $K[x]$, namely, $F = d_f^{-1} f$, where d_f is a generator of the fixed divisor of f .
- (3) An irreducible polynomial in $\text{Int}(D)$ is necessarily image-primitive.
- (4) In the equivalence class of f with respect to multiplication by non-zero constants in K , it is only the image-primitive elements that have a chance of being irreducible, or absolutely irreducible. We investigate for which $f \in D[x]$ an image-primitive $F \in \text{Int}(D)$ associated to f in $K[x]$ is absolutely irreducible.

Next, we remind the reader of the prime spectrum of $\text{Int}(R)$ where R is a discrete valuation domain and introduce some notation.

Notation 2.6. Let (R, M) be as in Convention 2.1. Further, let \hat{R} denote the M -adic completion of R and \hat{M} its maximal ideal.

- (1) For a monic, irreducible polynomial $q \in K[x]$, we write

$$Q_q = \text{Int}(R) \cap qK[x].$$

- (2) For $\alpha \in \hat{R}$, we write

$$M_\alpha = \{G \in \text{Int}(R) \mid G(\alpha) \in \hat{M}\}.$$

Here, $G(\alpha)$ is defined by extending the M -adically uniformly continuous function $G: R \rightarrow R$ uniquely to \hat{R} .

- (3) Note that, for $a \in R$,

$$M_a = \{G \in \text{Int}(R) \mid G(a) \in M\}$$

because $\hat{M} \cap K = M$.

(4) For $F \in \text{Int}(R)$, we write

$$\begin{aligned} \mathcal{M}(F) &= \{M_a \mid a \in R, F \in M_a\} \text{ and} \\ \mathcal{Q}(F) &= \{Q_q \mid q \text{ monic, irreducible in } K[x], F \in Q_q\}. \end{aligned}$$

Fact 2.7 ([4, Corollary V.1.2, Lemma V.1.3, Proposition V.2.2]). *Let (R, M) be as in Convention 2.1. Further, let \hat{R} denote the M -adic completion of R and \hat{M} its maximal ideal.*

Then

$$\text{Spec}(\text{Int}(R)) = \{(0)\} \cup \{Q_q \mid q \in K[x] \text{ monic, irreducible}\} \cup \{M_\alpha \mid \alpha \in \hat{R}\}.$$

2.2. Absolute irreducibility and prime separation. Again, we only introduce the notions and tools from factorization theory that we require below. For a thorough treatment of the topic, we refer to the textbook of Geroldinger and Halter-Koch [12].

Convention 2.8. Ring theoretic entities such as units, irreducible elements, prime ideals, etc. are defined with respect to the ring $\text{Int}(R)$, unless specified otherwise. Similarly, principal ideals: (F) means $F \text{Int}(R)$ unless specified otherwise.

Convention 2.9. As usual in factorization theory, we do not distinguish between associated elements of $\text{Int}(R)$, that is, elements that differ only by multiplication by a unit of $\text{Int}(R)$. (Recall that the units of $\text{Int}(R)$ are the units of R .) Also, regarding uniqueness, *essentially unique* means unique up to multiplication by units of $\text{Int}(R)$.

Definition 2.10. Let D be a domain. An irreducible element $d \in D$ is called *absolutely irreducible* if d^n factors uniquely in D for all integers $n \geq 1$.

Chapman and Krause [7, Lemma 2.1] showed for an irreducible element c of an atomic domain D the equivalence of the following two statements:

- (1) c is absolutely irreducible.
- (2) For all irreducible $b \in D$ which are not associated to c there exists a prime ideal P of D such that $b \in P$ and $c \notin P$.

We rewrite and refine this characterization below in Proposition 2.12. The proposition's detailed formulation in particular points out which prime ideals we need to consider in our further work.

Definition 2.11. For an ideal I of a domain D , let

$$V(I) = \{P \in \text{Spec}(D) \mid I \subseteq P\}$$

denote the subset of the prime spectrum of D of prime ideals containing I . For a principal ideal cD , we write $V(c)$ for $V(cD)$.

Recall that the radical \sqrt{I} of an ideal I is defined by

$$\sqrt{I} = \{d \in D \mid d^k \in I \text{ for some } k \in \mathbb{N}\}$$

and, as we all know, this is equivalent to

$$\sqrt{I} = \bigcap_{P \in V(I)} P.$$

Therefore, for ideals I and J , $\sqrt{I} \subseteq \sqrt{J}$ if and only if $V(I) \supseteq V(J)$. More generally, if for some class of ideals (e.g. finitely generated ideals), the radical is always an intersection of prime ideals from some special subset $\mathcal{S} \subseteq \text{Spec}(D)$, then again $\sqrt{I} \subseteq \sqrt{J}$ if and only if $V(I) \cap \mathcal{S} \supseteq V(J) \cap \mathcal{S}$.

This simple fact leads to the following extended version of the criterion of Chapman and Krause above.

Proposition 2.12. *Let D be an atomic domain and $\mathcal{S} \subseteq \text{Spec}(D)$ such that for every principal ideal cD , the radical \sqrt{cD} is the intersection of all $P \in \mathcal{S}$ containing c .*

For a non-zero non-unit $d \in D$, the following are equivalent.

- (1) d is **not** absolutely irreducible, i.e., some power d^n has a factorization into irreducibles essentially different from $d \cdots d$ (n copies of d).
- (2) Some power of d is divisible by some irreducible $c \in D$ that is not associated to d .
- (3) Some power of d is divisible by some non-unit $c \in D$ that is not associated to a power of d .
- (4) $d \in \sqrt{cD}$ for some non-unit $c \in D$ that is not associated to a power of d .
- (5) $\sqrt{dD} \subseteq \sqrt{cD}$ for some non-unit $c \in D$ that is not associated to a power of d .
- (6) $V(c) \subseteq V(d)$ for some non-unit $c \in D$ that is not associated to a power of d .
- (7) $V(c) \cap \mathcal{S} \subseteq V(d) \cap \mathcal{S}$ for some non-unit $c \in D$ that is not associated to a power of d .

Proof. (1) \Rightarrow (2): Suppose d^n has a factorization essentially different from $d \cdots d$ (n copies of d). Then, since $d^n = d^m$ for $n \neq m$ is impossible by cancellation in a domain, there exists an irreducible c not associated to d dividing d^n .

(2) \Rightarrow (1): Suppose $d^n = ce$ with c irreducible and not associated to d and $e \in D$. Then, since D is atomic, we can use a factorization $e = e_1 \cdots e_m$ into irreducibles to get a factorization $d^n = ce_1 \cdots e_m$ other than $d \cdots d$ (n copies of d).

For the remaining statements, the equivalence of each statement to the preceding one follows from elementary considerations. \square

Now we use specific information about the spectrum of $\text{Int}(R)$ to rephrase the proposition above for our setting.

Remark 2.13. Let (R, M) be as in Convention 2.1. We know that the radical \sqrt{I} of every finitely generated ideal I of $\text{Int}(R)$ is an intersection of maximal ideals of the special form

$$M_a = \{G \in \text{Int}(R) \mid G(a) \in M\}$$

for $a \in R$ and prime ideals of the form

$$Q_q = \text{Int}(R) \cap qK[x]$$

for monic, irreducible polynomials $q \in K[x]$. The maximal ideals of the form M_α for $\alpha \in \hat{R} \setminus R$ are redundant, for reasons of M -adic continuity.

Setting $\mathcal{S} = \mathcal{M}(f) \cup Q(f)$ in Proposition 2.12(7) gives the following.

Corollary 2.14. *Let (R, M) be as in Convention 2.1, and $F \in \text{Int}(R)$ a non-zero non-unit, and $\mathcal{M}(F)$ and $Q(F)$ as in Notation 2.6(4).*

- (1) F is absolutely irreducible in $\text{Int}(R)$ if and only if every non-unit $G \in \text{Int}(R)$ satisfying $\mathcal{M}(G) \cup Q(G) \subseteq \mathcal{M}(F) \cup Q(F)$ is associated to a power of F .
- (2) If there exists a non-unit $G \in \text{Int}(R)$ such that

$$\mathcal{M}(G) \cup Q(G) \subsetneq \mathcal{M}(F) \cup Q(F),$$

then F is not absolutely irreducible.

3. THE POSH SET OF A POLYNOMIAL

Let (R, M) be as in Convention 2.1. We will characterize absolutely irreducible polynomials that split over R in terms of their root sets. It is, therefore, convenient to work with a product of linear factors $f = \prod_{s \in S} (x - s)^{m_s}$ with $S \subseteq R$ and investigate whether the essentially unique image-primitive $F = d_f^{-1} f$ which is $K[x]$ -associated to f is absolutely irreducible, cf. Remark 2.5.

We now reformulate the criterion in Corollary 2.14 for absolute irreducibility in terms of f instead of F .

It is easily seen that $Q(F) = Q(f)$. The set $\mathcal{M}(F)$, however, is not invariant under multiplication with constants in K . To resolve this issue we now introduce a set $\mathcal{P}(F)$ that is in one-to-one correspondence with $\mathcal{M}(F)$ and is invariant under multiplication with constants in K , so that $\mathcal{P}(F) = \mathcal{P}(f)$.

Definition 3.1. Let (R, M) be as in Convention 2.1. For $F \in K[x]$, we denote by $\mathcal{P}(F)$ the *posh set* of F , that is,

$$\mathcal{P}(F) = \{r \in R \mid v(F(r)) > \min_{t \in R} v(F(t))\}.$$

Remark 3.2. Let (R, M) be as in Convention 2.1.

- (1) If $F \in \text{Int}(R)$ and d_F a generator of the fixed divisor of F (cf. Definition 2.2) then

$$\min_{t \in R} v(F(t)) = v(d_F).$$

- (2) A polynomial $F \in \text{Int}(R)$ is image-primitive if and only if there exists an element $a \in R$ such that $F(a) \notin M$. Therefore, F is image-primitive if and only if there exists $a \in R$ with $M_a \notin \mathcal{M}(F)$.

Remark 3.3. Let $F \in \text{Int}(R)$ be such that $\min_{r \in R} v(F(r)) = 0$, that is, F image-primitive. Then

$$a \in \mathcal{P}(F) \iff F(a) \in M.$$

There is, therefore, a one-to-one correspondence between $\mathcal{P}(F)$ and $\mathcal{M}(F)$, given by $a \mapsto M_a$, in other words,

$$\mathcal{M}(F) = \{M_a \mid a \in \mathcal{P}(F)\} \quad \text{and} \quad \mathcal{P}(F) = \{a \in R \mid M_a \in \mathcal{M}(F)\}.$$

Now, unlike $\mathcal{M}(F)$, the posh set of F is invariant under multiplication of F by non-zero constants in K . Suppose $f \in K[x]$ and $F \in \text{Int}(R)$ such that F is associated to f in $K[x]$, that is, $FK[x] = fK[x]$. If F is image-primitive then

$$\mathcal{M}(F) = \{M_a \mid a \in \mathcal{P}(f)\} \quad \text{and} \quad \mathcal{P}(F) = \{a \in R \mid M_a \in \mathcal{M}(F)\} = \mathcal{P}(f).$$

We can, therefore, let the posh set of f stand in for $\mathcal{M}(F)$ when we formulate a criterion for an image-primitive polynomial F to be absolutely irreducible. Before we formulate this criterion below in Proposition 3.5, we establish that by switching between F and the uniquely determined monic polynomial f which is $K[x]$ -associated to F we keep the information about the powers of these polynomials.

Lemma 3.4. Let (R, M) be as in Convention 2.1. Let $f, g \in K[x]$ be monic and $F, G \in \text{Int}(R)$ be image-primitive such that $FK[x] = fK[x]$ and $GK[x] = gK[x]$.

Then

$$g = f^n \iff G \approx F^n,$$

where \approx means associated in $\text{Int}(R)$, that is, differing only by multiplication by a unit of R .

Proof. Necessarily $G = c^{-1}g$ and $F = d^{-1}f$ for generators c and d of the fixed divisors of g and f , respectively. If $g = f^n$ then $c \approx d^n$ and, hence, $G = c^{-1}g \approx d^{-n}f^n = F^n$.

Conversely, if $G \approx F^n$, then g equals f^n because each is the unique monic generator of the ideal $GK[x] = F^nK[x]$. \square

Proposition 3.5. *Let (R, M) be as in Convention 2.1. Let $f \in K[x]$ be monic and $F \in \text{Int}(R)$ be image-primitive such that $FK[x] = fK[x]$.*

Then the following assertions are equivalent:

- (1) F is absolutely irreducible.
- (2) Every monic $g \in K[x]$ with $g \neq 1$ satisfying $Q(g) \subseteq Q(f)$ and $\mathcal{P}(g) \subseteq \mathcal{P}(f)$ is a power of f .

Proof. Assume (1). Let $G \in \text{Int}(R)$ be image-primitive with $GK[x] = gK[x]$. Then $Q(G) = Q(g) \subseteq Q(f) = Q(F)$ and

$$\mathcal{M}(G) = \{M_a \mid a \in \mathcal{P}(g)\} \subseteq \{M_a \mid a \in \mathcal{P}(f)\} = \mathcal{M}(F).$$

By Corollary 2.14, $G \approx F^n$ for some $n \in \mathbb{N}$, and, hence, $g = f^n$, by Lemma 3.4.

Conversely, assume (2). Now, if some $G \in \text{Int}(R)$ satisfies $\mathcal{M}(G) \subseteq \mathcal{M}(F)$ and $Q(G) \subseteq Q(F)$ then, since $\mathcal{M}(F) \subsetneq \{M_a \mid a \in R\}$, it follows that G is image-primitive, cf. Remark ???. Let g be monic with $gK[x] = GK[x]$. Then

$$\mathcal{P}(g) = \{a \in R \mid M_a \in \mathcal{M}(G)\} \subseteq \{a \in R \mid M_a \in \mathcal{M}(F)\} = \mathcal{P}(f)$$

and

$$Q(g) = Q(G) \subseteq Q(F) = Q(f).$$

By hypothesis, $g = f^n$ (for some n) follows, and, by Lemma 3.4, $G \approx F^n$. Applying Corollary 2.14, we conclude that F is absolutely irreducible. \square

Corollary 3.6. *Let (R, M) be as in Convention 2.1, $f \in K[x]$ be monic and $F \in \text{Int}(R)$ image-primitive, such that $fK[x] = FK[x]$.*

If there exists a monic, non-constant polynomial $g \in K[x]$ with

$$Q(g) \subseteq Q(f) \quad \text{and} \quad \mathcal{P}(g) \subsetneq \mathcal{P}(f)$$

or

$$Q(g) \subsetneq Q(f) \quad \text{and} \quad \mathcal{P}(g) \subseteq \mathcal{P}(f)$$

then F is not absolutely irreducible.

We now turn our attention to split integer-valued polynomials, which are our main subject of investigation.

Remark 3.7. Let $f = \prod_{s \in S} (x - s)^{m_s}$ with $m_s > 0$ positive integers for $s \in S$. For a monic, non-constant polynomial $g \in K[x]$ the condition

$$Q(g) \subseteq Q(f) \tag{2}$$

is equivalent to $g = \prod_{t \in T} (x - t)^{k_t}$ with $\emptyset \neq T \subseteq S$ and $k_t > 0$ for $t \in T$.

Of the two entities considered in Proposition 3.5 and Corollary 3.6, $Q(f)$ is completely determined by the irreducible factors of f , that is, in the case of a split polynomial, by the roots (no multiplicities considered). The posh set $\mathcal{P}(f)$ is also closely connected to the root set but here the multiplicities matter, as we shall see.

4. DISTRIBUTION OF THE ROOTS OF A SPLIT POLYNOMIAL

In this section we discuss the connection between the posh set of a split polynomial $f = \prod_{s \in S} (x - s)^{m_s}$ and the distribution of the root set S among the residue classes of powers of the maximal ideal M of the discrete valuation domain R .

Definition 4.1. Let (R, M) be as in Convention 2.1.

- (1) By an M -adic partition of R we understand a finite partition of R into residue classes of powers of M , that is

$$\mathcal{C} = \{s + M^{n_s} \mid s \in S\},$$

where $S \subseteq R$ is a finite set, $R = \bigcup_{s \in S} (s + M^{n_s})$ and $(s + M^{n_s}) \cap (t + M^{n_t}) = \emptyset$ for $s \neq t$. We say the set S is a *set of representatives of \mathcal{C}* .

- (2) The pair (S, \mathcal{C}) where \mathcal{C} is an M -adic partition and S is a set of representatives is also called a *pointed M -adic partition of R* .

Definition 4.2. Let (R, M) be as in Convention 2.1. We call $S \subseteq R$ (R, M) -balanced if, when we take for each $s \in S$ the minimal n_s such that $s + M^{n_s}$ contains no other element of S , the resulting disjoint basic M -adic neighborhoods $s + M^{n_s}$ cover R . If R and M are understood we just say *balanced* for (R, M) -balanced.

Remark 4.3. Note that the set of representatives S of a pointed M -adic partition (\mathcal{C}, S) is a balanced set; and conversely, that for each balanced set S the maximal basic M -adic neighborhoods $s + M^{n_s}$ disjoint from the remaining elements of S , together with S as a system of representatives, constitute a pointed M -adic partition of R .

The blocks of an M -adic partition can be visualized as the leaves of a q -adic tree where $q = |R/M|$. We obtain a balanced set by choosing one representative for each block. For example, Figure 1 shows the tree corresponding to the balanced set $\{0, 3, 6, 15, 24, 1, 2, 11, 20, 47, 128, 209, 74, 5, 8\}$ in \mathbb{Z} localized at 3.

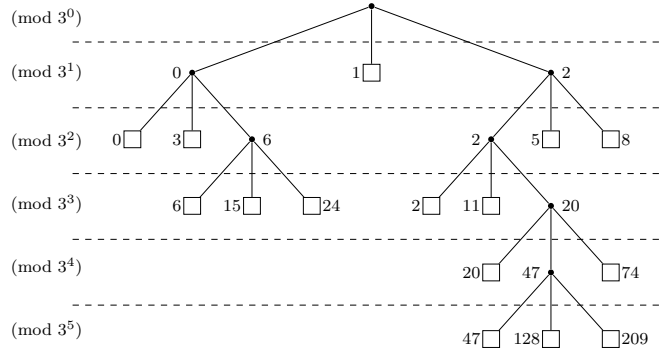


FIGURE 1.

Clearly not every finite set is balanced. We can, nevertheless, extend the argument of Remark 4.3 to associate a unique M -adic partition to every finite subset S of R in such a way that S contains a balanced subset associated to the same partition.

Lemma 4.4. Let (R, M) be as in Convention 2.1 and $S \subseteq R$ a finite subset of R . Then there exists a uniquely determined M -adic partition

$$\mathcal{C}_S = \{s + M^{n_s} \mid s \in S\}$$

of R such that every residue class $s + M^{n_s}$ that occurs as a block of \mathcal{C}_S contains both a residue class of M^{n_s+1} intersecting S and a residue class of M^{n_s+1} disjoint from S .

As a consequence, S contains a balanced subset S' with $\mathcal{C}_S = \mathcal{C}_{S'}$.

Proof. To see the existence of \mathcal{C}_S , we construct it, inductively, as follows. In each step k , we start with two sets of M -adic neighborhoods: \mathcal{C}_k (containing residue classes of M^n for various $n < k$ that have already been chosen as blocks of our partition), and \mathcal{B}_k (containing residue classes of M^k under consideration as potential blocks of the partition), such that

- (1) $\mathcal{C}_k \uplus \mathcal{B}_k$ is a partition of R (we use \uplus to denote disjoint unions)
- (2) each $(r + M^n) \in \mathcal{C}_k$ contains both a residue class of M^{n+1} intersecting S and a residue class of M^{n+1} disjoint from S
- (3) $(r + M^k) \cap S \neq \emptyset$ for each $(r + M^k) \in \mathcal{B}_k$.

The process terminates when $\mathcal{B}_k = \emptyset$ and, consequently, $\mathcal{C}_k = \mathcal{C}_S$ is a partition of R with the desired properties.

At the beginning of Step 0, $\mathcal{B}_0 = \{R\}$ and $\mathcal{C}_0 = \emptyset$. If some residue classes of M contain elements of S and some do not, then we set $\mathcal{C}_1 = \{R\}$ and $\mathcal{B}_1 = \emptyset$ and we are done.

Otherwise, we break up R into residue classes of M and put these in \mathcal{B}_1 , so that we have $\mathcal{C}_1 = \emptyset$ and $\mathcal{B}_1 = \{r_1 + M, \dots, r_q + M\}$.

At Step k , we define \mathcal{C}_{k+1} to be the union of \mathcal{C}_k and the set of those residue classes of M^k in \mathcal{B}_k that contain residue classes of M^{k+1} intersecting S as well as residue classes of M^{k+1} disjoint from S . The remaining residue classes of M^k in \mathcal{B}_k , containing only such residue classes of M^{k+1} that intersect S nontrivially, we split into residue classes of M^{k+1} and let \mathcal{B}_{k+1} be the set of these residue classes.

Since S is finite, the process terminates and we get the desired partition of R .

To see uniqueness, consider that each residue class $r + M^k$ that we add to \mathcal{C}_k , and, eventually, to \mathcal{C}_S , in Step k must occur as a block of the partition, because otherwise \mathcal{C}_S would have to contain as blocks some residue classes contained in $r + M^k$ that are disjoint from S . \square

It is a key fact in our characterization of split absolutely irreducible polynomials that their posh set is as small as can be. To formalize this, we introduce the rich set of a split polynomial, which is always contained in its posh set, and show that the posh set equals the rich set in the case of an absolutely irreducible split polynomial.

Definition 4.5. Let (R, M) be as in Convention 2.1 and $S \subseteq R$ a finite subset.

- (1) We call the (uniquely determined) partition \mathcal{C}_S of Lemma 4.4 the *partition associated to S* .
- (2) For $s \in S$, let $\rho_S(s) \in \mathbb{N}$ be the uniquely determined non-negative integer such that $s + M^{\rho_S(s)}$ is a partition block of \mathcal{C}_S .
- (3) An *S -rich neighborhood* is a residue class $s + M^{\rho_S(s)+1}$ with $s \in S$.
- (4) An *S -poor neighborhood* is a residue class of the form $r + M^{\rho_S(s)+1}$ disjoint from S where $r \in s + M^{\rho_S(s)}$ for some $s \in S$.
- (5) The *rich set of S* , denoted by $\mathcal{R}(S)$, is the union of the rich neighborhoods of the partition \mathcal{C} associated to S , that is

$$\mathcal{R}(S) = \bigcup_{s \in S} s + M^{\rho_S(s)+1}.$$

- (6) For a polynomial $f \in K[x]$ that splits over R the *rich set of f* , denoted by $\mathcal{R}(f)$, is defined to be the rich set of the set of its roots.

Remark 4.6.

- (1) It follows from the proof of Lemma 4.4 that, for $s \in S$, $\rho_S(s) \in \mathbb{N}_0$ is the minimal number such that $s + M^{\rho_S(s)}$ contains an S -rich neighborhood as well as an S -poor neighborhood. In particular, if $T \subseteq S$ is a subset of S , then $\rho_T(t) \leq \rho_S(t)$ for all $t \in T$.
- (2) For $n \in \mathbb{N}$, the following implication holds

$$n \leq \rho_S(s) \implies s + M^n \not\subseteq \mathcal{R}(S).$$

In particular, if $T \subseteq S$ and there exists $t \in T$ with $\rho_T(t) < \rho_S(t)$, then $\mathcal{R}(T) \not\subseteq \mathcal{R}(S)$.

- (3) If S is a balanced set then the rich set

$$\mathcal{R}(S) = \bigsqcup_{s \in S} s + M^{\rho_S(s)+1}$$

is the disjoint union of the S -rich neighborhoods $s + M^{\rho_S(s)+1}$.

- (4) If S' is a balanced set contained in S with $\mathcal{C}_S = \mathcal{C}_{S'}$, then $\rho_{S'}(s) = \rho_S(s)$ for all $s \in S'$ and $\mathcal{R}(S') \subseteq \mathcal{R}(S)$. Note that equality may hold even if S is not balanced as an S -rich neighborhood can contain more than one element of S .

Definition 4.7. Let (R, M) be as in Convention 2.1 and $S \subseteq R$ a finite subset. We call a set S' a *balanced set associated to S* if S' is balanced and contained in S with $\mathcal{C}_S = \mathcal{C}_{S'}$. Further, for $t \in S'$, we write

$$S_t = \{s \in S \mid s + M^{\rho_S(s)} = t + M^{\rho_S(t)}\}$$

for the set of all elements in S which are elements of the partition block of $t \in S'$ of the partition $\mathcal{C}_S = \mathcal{C}_{S'}$.

We are ready to show that the rich set is always contained in the posh set of a split polynomial. Let $f \in K[x]$ be a split monic polynomial whose root set S is a subset of R .

As the posh set of f is invariant under multiplication of f by non-zero constants and the rich set of f only depends on the set of roots S , we may assume that

$$f = \prod_{s \in S} (x - s)^{h_s} \in R[x]$$

where $S \subseteq R$ is finite and $h_s \in \mathbb{N}$ for $s \in S$. Let $S' \subseteq S$ be a balanced set associated to S , cf. Definition 4.7.

Let $u, t \in S'$ with $u \neq t$ and $w \in u + M^{\rho_S(u)}$ and $s \in t + M^{\rho_S(t)}$ be elements in two distinct partition blocks of the partition $\mathcal{C}_S = \mathcal{C}_{S'}$. Then the valuation $v(w - s)$ only depends on the blocks and not the specific choice of elements w and s , that is,

$$v(w - s) = v(u - t) \text{ for all } w \in u + M^{\rho_S(u)}, s \in t + M^{\rho_S(t)} \text{ with } u, t \in S', u \neq t. \quad (3)$$

Therefore, if $w \in u + M^{\rho_S(u)}$ for $u \in S'$, then

$$\begin{aligned} v(f(w)) &= \sum_{t \in S} h_t v(w - t) = \sum_{t \in S'} \sum_{s \in S_t} h_s v(w - s) \\ &= \sum_{s \in S_u} v(w - s) h_s + \sum_{\substack{t \in S' \\ t \neq u}} v(u - t) \left(\sum_{s \in S_t} h_s \right). \end{aligned} \quad (4)$$

Observe that the second summand in the last line of Equation (4) only depends on the block $u + M^{\rho_S(u)}$, not the specific choice of w .

For the first summand, however, it makes a significant difference whether w is element of an S -rich or an S -poor neighborhood. If w is in an S -poor neighborhood

of the block $u + M^{\rho_S(u)}$, then

$$v(w - s) = v(w - u) = \rho_S(u) \text{ for all } s \in S_u. \quad (5)$$

Now, if r is in an S -rich neighborhood of the partition block of $u + M^{\rho_S(u)}$, then the following hold:

- $v(r - s) \geq \rho_S(u)$ for all $s \in S_u$ and
- there exists $s \in S \cap r + M^{\rho_S(s)+1}$ such that $v(r - s) > \rho_S(u)$.

Therefore, for this choice of w and r , it follows that

$$v(f(r)) - v(f(w)) = \sum_{s \in S_u} (v(r - s) - \rho_S(u)) h_s > 0$$

and hence

$$v(f(r)) > v(f(w)) \geq \min_{t \in R} v(f(t)) = v(d_f) \quad (6)$$

where d_f is a generator of the fixed divisor of f . This immediately implies the following lemma

Lemma 4.8. *Let (R, M) be as in Convention 2.1, $S \subseteq R$ a finite set and $f = \prod_{s \in S} (x - s)^{m_s}$ with $m_s \in \mathbb{N}$ for $s \in S$.*

Then every element in an S -rich neighborhood of the partition \mathcal{C}_S is in the posh set $\mathcal{P}(f)$, that is, $\mathcal{R}(f) \subseteq \mathcal{P}(f)$.

In other words, every element $w \in R$ with $v(f(w)) = \min_{t \in R} v(f(t))$ is in an S -poor neighborhood of \mathcal{C}_S .

5. CHARACTERIZING SPLIT POLYNOMIALS WITH (RELATIVELY) SMALL POSH SETS

There are two ways to characterize small posh sets. On one hand, the inclusion $\mathcal{R}(f) \subseteq \mathcal{P}(f)$ (Lemma 4.8) means that $\mathcal{P}(f)$ is smallest possible if $\mathcal{P}(f) = \mathcal{R}(f)$ (Lemma 5.7). On the other hand, we can measure the size of the posh set by a finitely additive probability measure (Lemma 5.3).

Definition 5.1. For an ideal I of finite index in R let $\|I\| = [R : I]$ and let σ be the finitely additive probability measure on R defined by the requirement

$$\sigma(r + I) = \frac{1}{\|I\|}$$

whenever $r \in R$ and I is an ideal of finite index in R ; cf. [11]. For our purposes, all we need to know about σ is the values that it takes on finite unions of residue classes of ideals of finite index.

Remark 5.2. Let (R, M) be a local ring with finite residue field R/M of order q .

- (1) If $S \subseteq R$ is a finite set and $n_s \in \mathbb{N}$ for $s \in S$, then

$$\sigma\left(\bigsqcup_{s \in S} s + M^{n_s}\right) = \sum_{s \in S} \frac{1}{q^{n_s}}.$$

- (2) If $A \subseteq B$ are subsets of R that are each a finite union of residue classes of powers of M , then $A = B$ if and only if $\sigma(A) = \sigma(B)$.

Next, we discuss the question under which conditions the rich set and the posh set have minimal σ -measure.

Lemma 5.3. *Let (R, M) be as in Convention 2.1 and $|R/M| = q$. Let $f \in K[x]$ such that f splits over K whose root set S is a subset of R .*

Then

- (1) $\mathcal{R}(f) = \mathcal{P}(f)$ if and only if $\sigma(\mathcal{R}(f)) = \sigma(\mathcal{P}(f))$.

- (2) $\sigma(\mathcal{R}(f)) = \sigma(\mathcal{R}(S)) \geq \frac{1}{q}$. Equality holds if and only if every block of \mathcal{C}_S contains only one rich neighborhood.
- (3) $\sigma(\mathcal{P}(f)) = \frac{1}{q}$ if and only if $\mathcal{R}(f) = \mathcal{P}(f)$ and every block of \mathcal{C}_S contains only one rich neighborhood.

Proof. Recall that $\mathcal{R}(f) \subseteq \mathcal{P}(f)$ by Lemma 4.8.

Ad (1). $\mathcal{P}(f)$ and $\mathcal{R}(f)$ are each a finite union of residue classes of various powers of M , and so is $\mathcal{P}(f) \setminus \mathcal{R}(f)$. For sets of this kind, σ takes a positive value unless they are empty.

Ad (2). Each block $C = z + M^n$ of the partition \mathcal{C}_S of R contains at least one S -rich neighborhood $s + M^{n+1}$ and therefore $\sigma(\mathcal{R}(f) \cap C) \geq \frac{1}{q^{n+1}} = \frac{1}{q}\sigma(C)$. It follows that

$$\sigma(\mathcal{R}(f)) = \sum_{C \in \mathcal{C}_S} \sigma(\mathcal{R}(f) \cap C) \geq \frac{1}{q} \sum_{C \in \mathcal{C}_S} \sigma(C) = \frac{1}{q}.$$

By the same token, $\sigma(\mathcal{R}(f)) = \frac{1}{q}$ if and only if $\sigma(\mathcal{R}(f) \cap C) = \frac{1}{q}\sigma(C)$ for every block of the partition. Since every block contains at least one rich neighborhood, this is equivalent to saying that every block of the partition contains exactly one rich neighborhood.

(3) now follows from (1) and (2). \square

Remark 5.4. Balanced sets have two properties:

- (1) The rich set $\mathcal{R}(S)$ of a balanced set has the minimal possible σ -measure. This is equivalent to each block of the associated partition having exactly one rich neighborhood, see Lemma 5.3(2).
- (2) Every balanced set is minimal with respect to inclusion among all finite sets sharing the same rich set. This property is equivalent to every rich neighborhood containing only one element of the underlying set.

Note that balanced sets are characterized by these two properties.

We now characterize the case where $\mathcal{R}(f) = \mathcal{P}(f)$ holds in terms of root multiplicities. For this purpose, we revisit the observations made before Lemma 4.8 and recall some notation.

Namely, suppose $S \subseteq R$ is a finite (not necessarily balanced) set, S' a balanced set associated to S and for $t \in S'$ let $S_t = \{s \in S \mid s + M^{\rho_S(s)} = t + M^{\rho_S(t)}\}$, cf. Definition 4.7. Further, let $f = \prod_{s \in S} (x - s)^{h_s}$ where h_s is a positive integer for each $s \in S$. If $w \in R$ is an element of an S -poor neighborhood of the partition block $u + M^{\rho_S(u)}$ of \mathcal{C}_S , then by Equations (4) and (5), it follows that

$$v(f(w)) = \rho_S(u) \sum_{s \in S_u} h_s + \sum_{u \neq t \in S'} v(u - t) \left(\sum_{s \in S_t} h_s \right). \quad (7)$$

Note that the right hand side of this equality does not depend on the specific choice of w , that is, $v(f(w))$ is the same value for all $w \in u + M^{\rho_S(u)} \setminus \mathcal{R}(f)$.

For $t \in S'$, let $m_t = \sum_{s \in S_t} h_s$ denote the number of roots s of f (counting multiplicities) which are elements of the block $t + M^{\rho_S(s)}$. Then, by Equation (7), the column vector $(m_s)_{s \in S'}$ is a solution over the positive integers to the linear equation system

$$\sum_{\substack{t \in S' \\ t \neq u}} v(t - u)x_t + \rho_S(u)x_u = v(f(w)) \text{ with } u \in S', \quad (8)$$

where $w \in u + M^{\rho_S(u)} \setminus \mathcal{R}(f)$ is fixed, but arbitrary. This motivates the following definition.

Definition 5.5. Let \mathcal{C} be an M -adic partition of R and T a set of representatives of \mathcal{C} , that is,

$$\mathcal{C} = \mathcal{C}_T = \{s + M^{\rho_T(s)} \mid s \in T\}$$

as in Definition 4.1. We define the *partition matrix* of \mathcal{C} to be the square matrix $A = A_{\mathcal{C}}$ whose rows and columns are indexed by \mathcal{C} , or, equivalently, by T : $A_{\mathcal{C}} = (a_{s,t})_{s,t \in T}$, where

$$a_{s,t} = \begin{cases} \rho_T(s) & s = t \\ v(s-t) & s \neq t \end{cases}.$$

Note that $A_{\mathcal{C}}$ depends only on \mathcal{C} , not on the system T of representatives.

Recall that $\mathcal{R}(f) = \mathcal{P}(f)$ holds if and only if for each $w \in R$ which is an element of an S -poor neighborhood

$$v(f(w)) = \min_{t \in R} v(f(t))$$

holds. Considering that $\min_{t \in R} v(f(t)) = v(d_f)$, this means that the right hand side of the linear equation system (8) is the vector $(v(d_f))_{s \in S'}$, whose every coordinate is the same. We introduce the following notation to make it easier to refer to a vector of this form.

Notation 5.6. For $n \in \mathbb{N}$, let $\mathbf{1}_n = (1)_{1 \leq i \leq n}$ denote the vector whose every entry equals 1. Moreover, we write $\mathbf{x} = (x_i)_{1 \leq i \leq n}$ for a vector of indeterminates where we omit the size n for better readability.

Lemma 5.7. *Let (R, M) be as in Convention 2.1 and*

$$f = \prod_{s \in S} (x - s)^{h_s}$$

for a finite subset $S \subseteq R$ and positive integers h_s for $s \in S$.

Further, let $A = A_{\mathcal{C}_S}$ be the partition matrix of \mathcal{C}_S , $S' \subseteq S$ be a balanced set associated to S and $n = |S'| = |\mathcal{C}_S|$.

If $m_t = \sum_{s \in S_t} h_s$ denotes the number of roots of f (counted with multiplicities) in the partition block $t + M^{\rho_S(t)}$ for $t \in S'$ (cf. Definition 4.7), then the following assertions are equivalent:

- (1) $\mathcal{P}(f) = \mathcal{R}(f)$.
- (2) For every r in an S -poor neighborhood, $v(f(r)) = v(d_f)$.
- (3) For every r in an S -poor neighborhood, $v(f(r))$ is the same value.
- (4) The column vector $(m_t)_{t \in S'}$ is a solution to $A\mathbf{x} = v(d_f)\mathbf{1}_n$.
- (5) The column vector $(m_t)_{t \in S'}$ is a solution to $A\mathbf{x} = e\mathbf{1}_n$ for some $e \in \mathbb{N}_0$ where $e = 0$ if and only if $n = 1$.

Proof. Recall $\mathcal{P}(f) = \{s \in R \mid v(f(s)) > v(d_f)\}$ and that $\mathcal{R}(f) \subseteq \mathcal{P}(f)$ holds by Lemma 4.8(1). Moreover, $v(d_f) = v(f(w))$ for some $w \in R$ which is contained in an S -poor neighborhood. Also, $|S'| > 1$ if and only if \mathcal{C}_S contains more than one partition block. In this case S contains a complete set of residues modulo M which is equivalent to $v(d_f) > 0$.

Now, $\mathcal{P}(f) = \mathcal{R}(f)$ if and only if no S -poor neighborhood is contained in the posh set $\mathcal{P}(f)$, meaning, for every w in an S -poor neighborhood, $v(f(w)) = v(d_f)$. This is the case if and only if $v(f(w))$ is the same for all elements w of S -poor neighborhoods. Since by Equation (8)

$$\sum_{t \in S'} a_{u,t} m_t = v(f(w))$$

holds for every w in an S -poor neighborhood of the block $u + M^{\rho_S(u)}$ with $u \in S'$, the result follows. \square

Given a split polynomial f , Lemma 5.7 provides us with an easy computational method to check whether $\mathcal{R}(f) = \mathcal{P}(f)$ holds. In addition, we can also use the equivalent assertion of the lemma to construct an absolutely irreducible polynomial whose root set is a given balanced set.

6. CONSTRUCTING SPLIT ABSOLUTELY IRREDUCIBLE POLYNOMIALS

The goal of this section is to prove Theorem 1. Given a balanced set S we construct a (uniquely determined) absolutely irreducible polynomial whose root set is S , by computing root multiplicities $(m_s)_{s \in S}$ such that $f = \prod_{s \in S} (x-s)^{m_s}$ satisfies a condition which by Proposition 3.5 implies that $d_f^{-1} f$ is absolutely irreducible.

For this we show that for a balanced set S there exists a vector of multiplicities such that the corresponding polynomial satisfies condition (3) of Lemma 5.7. We then prove that the system matrix of the linear equation system in condition (5) of the same lemma is non-singular. In the last part of the section we show that the solution over the positive integers with minimal possible right-hand side e is the right choice for the multiplicities of the roots.

6.1. Choosing the multiplicities of the roots.

Proposition 6.1. *Let (R, M) be as in Convention 2.1, S a balanced set and \mathcal{C}_S the partition associated to it (Definition 4.5).*

Then there exists a vector $(m_s)_{s \in S} \in \mathbb{N}^{|S|}$ of positive integers such that $f = \prod_{s \in S} (x-s)^{m_s}$ satisfies: for every r in an S -poor neighborhood of \mathcal{C}_S , $v(f(r))$ takes the same value e .

Proof. Let m be maximal such that some residue class of M^m is a block of \mathcal{C}_S . By (reverse) induction from $n = m$ down to $n = 0$ we show:

For every residue class C of M^n that is a union of blocks of \mathcal{C}_S we can find multiplicities k_s for all $s \in S \cap C$ such that $f_C = \prod_{s \in S \cap C} (x-s)^{k_s}$ satisfies: for every r in a poor neighborhood of \mathcal{C}_S contained in C , $v(f_C(r))$ takes the same value. The statement for $n = 0$ proves the lemma, since then $C = R$.

For $n = m$, $C \cap S$ contains a single element s , and $k_s = 1$ works. Now let C be a residue class of M^n that is a union of blocks of \mathcal{C}_S . Either C itself is a block of \mathcal{C}_S (and we can set $k_s = 1$ for the single element s in $S \cap C$), or C is the disjoint union of C_i for $1 \leq i \leq q$, each C_i a residue class of M^{n+1} that is a union of blocks of \mathcal{C}_S .

In this case, we may assume, by induction hypothesis, that we have assigned a multiplicity k_s to each $s \in S \cap C_i$ such that for each i , the polynomial

$$f_i = \prod_{s \in S \cap C_i} (x-s)^{k_s}$$

satisfies: for every r in a poor neighborhood of \mathcal{C}_S contained in C_i , $v(f_i(r))$ takes the same value, say, a_i .

Further, note that, for each t in any C_j with $j \neq i$, $v(f_i(t))$ takes the same value, say e_i , with $e_i < a_i$. We set $d_i = a_i - e_i$ for $1 \leq i \leq q$.

Now let $c = \text{lcm}(d_i \mid 1 \leq i \leq q)$ and $c_i = \frac{c}{d_i}$. For $s \in S \cap C_i$, set $h_s = c_i k_s$. Then $f_C = \prod_{s \in S \cap C} (x-s)^{h_s} = \prod_{i=1}^q f_i^{c_i}$ satisfies that $v(f_C(r))$ takes the same value for all r in any poor neighborhood of \mathcal{C}_S contained in C .

Indeed, if r is in a poor neighborhood of \mathcal{C}_S contained in C_i , then $v(f_C(r)) = \sum_{j=1}^q c_j v(f_j(r)) = c_i a_i + \sum_{j \neq i} c_j e_j$, where $c_i a_i = c_i (d_i + e_i) = c + c_i e_i$ and hence $v(f_C(r)) = c + \sum_{j=1}^q c_j e_j$, which does not depend on i . \square

6.2. The partition matrix is non-singular. We assume here that S is a balanced set with $|S| > 1$, or equivalently, the partition \mathcal{C}_S contains more than one block. (Otherwise, the partition matrix is (0). We treat this case separately in the proof of Theorem 1 and Proposition 6.7.)

Lemma 6.2. *Let $u_1, u_2, \dots, u_n, b \in \mathbb{Q}_{>0}$ be positive rationals such that $u_i > b$ for all $1 \leq i \leq n$ and let*

$$A = \begin{pmatrix} u_1 & b & \cdots & b \\ b & u_2 & \cdots & b \\ \vdots & & \ddots & \vdots \\ b & b & \cdots & u_n \end{pmatrix}.$$

Then $\det(A) > 0$.

Proof. We prove by induction on n . If $n = 1$, then $\det(A) = u_1 > 0$ which proves the basis.

For $n \geq 2$, we eliminate the off-diagonal entries in the first column by adding the $\frac{b}{u_1}$ -fold of the first row to all other rows. The resulting matrix is

$$\begin{pmatrix} u_1 & b & \cdots & b \\ 0 & \hat{A} & & \end{pmatrix}$$

where

$$\hat{A} = \begin{pmatrix} u_2 - \frac{b^2}{u_1} & \cdots & b - \frac{b^2}{u_1} \\ & \ddots & \vdots \\ b - \frac{b^2}{u_1} & \cdots & u_n - \frac{b^2}{u_1} \end{pmatrix}.$$

Since $u_i > b > 0$ for all $1 \leq i \leq n$, it follows that

$$u_i - \frac{b^2}{u_1} > b - \frac{b^2}{u_1} > 0.$$

Therefore \hat{A} satisfies the assumptions of the lemma and by induction hypothesis it follows that $\det(\hat{A}) > 0$ and

$$\det(A) = u_1 \det(\hat{A}) > 0.$$

□

Notation 6.3. For $n, t \in \mathbb{N}$, we write $\mathbf{E}_{n,t} = (1)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq t}}$ for the $(n \times t)$ -matrix all of whose entries are 1. Note that $\mathbf{1}_n = \mathbf{E}_{n,1}$.

Lemma 6.4. *Let $A \in \mathbb{Q}^{n \times n}$ such that the equation system $A\mathbf{x} = \mathbf{1}_n$ has a solution over the positive rationals.*

If $\det(A) \neq 0$, then $\det(A + \mathbf{E}_{n,n}) \neq 0$.

Proof. Let $\mathbf{u} = (u_i)_{1 \leq i \leq n} \in \mathbb{Q}_{>0}^n$ with $A\mathbf{u} = \mathbf{1}_n$ and let \mathbf{a}_j denote the j -th column of A for $1 \leq j \leq n$, that is, $\sum_{j=1}^n u_j \mathbf{a}_j = \mathbf{1}_n$.

Let $\lambda_1, \dots, \lambda_n \in \mathbb{Q}$ such that

$$0 = \sum_{j=1}^n \lambda_j (\mathbf{a}_j + \mathbf{1}_n) = \sum_{j=1}^n \lambda_j \mathbf{a}_j + \left(\sum_{i=1}^n \lambda_i \right) \mathbf{1}_n \quad (9)$$

$$= \sum_{j=1}^n \lambda_j \mathbf{a}_j + \left(\sum_{i=1}^n \lambda_i \right) \left(\sum_{j=1}^n u_j \mathbf{a}_j \right) \quad (10)$$

$$= \sum_{j=1}^n \left(\lambda_j + u_j \sum_{i=1}^n \lambda_i \right) \mathbf{a}_j \quad (11)$$

Since $\det(A) \neq 0$, it follows that

$$\lambda_j + u_j \sum_{i=1}^n \lambda_i = \lambda_j(1 + u_j) + u_j \sum_{\substack{i=1 \\ i \neq j}}^n \lambda_i = 0$$

for all $1 \leq j \leq n$. In other words, $(\lambda_1, \dots, \lambda_n)$ is a solution of the homogeneous linear equation system $U\mathbf{x} = 0$ where

$$U = \begin{pmatrix} u_1 + 1 & u_1 & \cdots & u_1 \\ u_2 & u_2 + 1 & \cdots & u_2 \\ \vdots & & \ddots & \vdots \\ u_n & u_n & \cdots & u_n + 1 \end{pmatrix}.$$

Since $u_i > 0$ for $1 \leq i \leq n$ and

$$\det(U) = u_1 u_2 \cdots u_n \cdot \det \begin{pmatrix} 1 + \frac{1}{u_1} & 1 & \cdots & 1 \\ 1 & 1 + \frac{1}{u_2} & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 + \frac{1}{u_n} \end{pmatrix}$$

it follows from Lemma 6.2 that $\det(U) \neq 0$. We conclude that $\lambda_1 = \cdots = \lambda_n = 0$ and therefore the columns of $A + \mathbf{E}_{n,n}$ are linearly independent. \square

Proposition 6.5. *Let (R, M) be as in Convention 2.1. Further, let \mathcal{C} be an M -adic partition of R consisting of more than one block and $A_{\mathcal{C}}$ the partition matrix of \mathcal{C} . Then $\det(A_{\mathcal{C}}) \neq 0$.*

Proof. Let S denote a set of representatives of the blocks of \mathcal{C} . Then S is balanced, $|S| > 1$ by assumption, $\mathcal{C} = \mathcal{C}_S$ and the S can be used as index set for the rows and columns of A .

Let $n \in \mathbb{N}$ be maximal such that $s + M^n$ is a partition block of \mathcal{C} . We prove the assertion by induction on n . If $n = 1$, then S is a set of representatives of the residue classes of R modulo M , $a_{s,s} = 1$ and $a_{s,t} = 0$ for all $s \neq t \in S$. Hence A is the identity matrix and $\det(A) > 0$.

Now let $n \geq 2$. As index set of the rows and columns we assume that S is endowed with a fixed linear ordering of its elements in such a way that blocks contained in the same residue class of M are adjacent. This makes the matrix a block diagonal matrix with blocks B_1, \dots, B_q each block belonging to one residue class of M . Since $\det(A) = \det(B_1) \cdots \det(B_q)$, it suffices to show that the determinants of each block are positive.

We fix i and set $B = B_i$ and let s be the size of B and $\tilde{S} \subseteq S$ the subset of S that serves as index set for B . If $|\tilde{S}| = 1$, then $B = (1)$ and hence $\det(B) \neq 0$. So assume that $|\tilde{S}| > 1$.

We now construct a new partition $\tilde{\mathcal{P}}$ of R by (bijectively) mapping the elements of \tilde{S} to a set of representatives T of $\tilde{\mathcal{C}}$. We do this as follows. Let a_1, \dots, a_q be a set of representatives of R modulo M and let $t \in R$ with $M = tR$ be a uniformizer of R . Then for each $n \in \mathbb{N}$ and each residue class of M^n there is a uniquely determined representative of the form

$$\sum_{j=0}^{n-1} a_{i_j} t^j \text{ with } 1 \leq i_0, \dots, i_{n-1} \leq q.$$

Without restriction, let a_1 be the representative modulo M of all the elements in \tilde{S} . We define the map

$$\ell: \bigcup_{n \geq 1} R/M^n \rightarrow \bigcup_{n \geq 0} R/M^n$$

$$\left(a_1 + \sum_{j=1}^{n-1} a_{i_j} t^j \right) + M^n \mapsto \left(\sum_{j=1}^{n-1} a_{i_j} t^{j-1} \right) + M^{n-1}$$

which bijectively maps the residue classes of R modulo M^n with $n > 1$ contained in $a_1 + M$ to all residue classes of R modulo M^{n-1} . For each $s \in \tilde{S}$, let $t(s) \in R$ such that

$$\ell\left(s + M^{\rho_S(s)}\right) = t(s) + M^{\rho_S(s)-1}$$

and set

$$T = \{t(s) \mid s \in \tilde{S}\}.$$

Since \mathcal{C}_S is a partition of R , it follows that

$$a_1 + M = \bigcup_{s \in \tilde{S}} s + M^{\rho_S(s)} \quad (12)$$

and hence

$$\tilde{\mathcal{P}} = \left\{ \ell\left(s + M^{\rho_S(s)}\right) \mid s \in \tilde{S} \right\} = \left\{ t + M^{\rho_S(s)-1} \mid t \in T \right\}$$

is an M -adic partition of R . By construction, T is a set of representatives of $\tilde{\mathcal{P}}$ and T is a balanced set with $|T| = |\tilde{S}| > 1$. Moreover, $\rho_T(t(s)) = \rho_S(s) - 1 = b_{s,s} - 1$ and $v(t(s) - t(\tilde{s})) = v(s - \tilde{s}) - 1 = b_{s,\tilde{s}} - 1$ for all $s, \tilde{s} \in \tilde{S}$.

Therefore, $B - \mathbf{E}_{|T|,|T|}$ is the partition matrix of $\tilde{\mathcal{P}}$ and since $\max_{t \in T} \rho_T(t) < \max_{s \in \tilde{S}} \rho_S(s)$ it follows by induction that $\det(B - \mathbf{E}_{|T|,|T|}) \neq 0$.

Moreover, by Proposition 6.1, there exists a positive solution vector to the system $(B - \mathbf{E}_{|T|,|T|})\mathbf{x} = \mathbf{1}_s$. Hence $\det(B) \neq 0$ according to Lemma 6.4 which completes the proof. \square

6.3. The equalizing polynomial.

Definition 6.6. An integer vector $(m_i)_{1 \leq i \leq n} \in \mathbb{Z}^n$ is called *unimodular* if

$$\gcd(m_i \mid 1 \leq i \leq n) = 1.$$

Proposition 6.7. Let (R, M) be as in Convention 2.1, S be a balanced subset of R , $n = |S|$ and $A_{\mathcal{C}_S}$ the partition matrix of the partition \mathcal{C}_S associated to S .

Then there exists a uniquely determined unimodular solution $(m_s)_{s \in S} \in \mathbb{N}^n$ of $A_{\mathcal{C}_S} \mathbf{x} = e \mathbf{1}_n$ with $e \in \mathbb{N}_0$.

In addition, if $(k_s) \in \mathbb{N}^n$ is a solution to $A_{\mathcal{C}_S} \mathbf{x} = \tilde{e} \mathbf{1}_n$ with $\tilde{e} \in \mathbb{N}_0$, then $k_s = \ell m_s$ for all $s \in S$ and $\tilde{e} = \ell e$ for some $\ell \in \mathbb{N}$.

Proof. By Proposition 6.1, there exist $(m_s)_{s \in S}$ such that $f = \prod_{s \in S} (x - s)^{m_s}$ is a polynomial for which $v(f(w)) = e \in \mathbb{N}_0$ is the same for each $w \in R \setminus \mathcal{R}(f)$. Therefore, by Lemma 5.7, $(m_s)_{s \in S}$ is a solution of the linear equation system $A_{\mathcal{C}_S} \mathbf{x} = e \mathbf{1}_n$. Common integer factors of the coordinates of $(m_s)_{s \in S}$ necessarily divide e . By cancelling them out we can assume that $(m_s)_{s \in S}$ is a unimodular vector (and $e \in \mathbb{N}_0$).

If $n = 1$, then $A_{\mathcal{C}_S} = (0)$ which implies that $e = 0$ and (1) is the uniquely determined unimodular integer solution to the equation system. The second assertion of the proposition immediately follows.

From now on we assume that $n > 1$ and hence $e > 0$ (see Lemma 5.7(5)). Further, let $(k_s) \in \mathbb{N}^n$ is a solution vector to $A_{\mathcal{C}_S} \mathbf{x} = \tilde{e} \mathbf{1}_n$ with $\tilde{e} \in \mathbb{N}_0$.

Let $c = \gcd(e, \tilde{e})$ and write $e = cd$ and $\tilde{e} = c\ell$ for suitable (coprime) positive integers d and ℓ . Then

$$A_{P_S} \left(\frac{m_s}{d} \right)_{s \in S} = c\mathbf{1}_n = A_{P_S} \left(\frac{k_s}{\ell} \right)_{s \in S}.$$

By assumption $n > 1$ and matrix A_C has non-zero determinant by Proposition 6.5. Therefore $\left(\frac{m_s}{d} \right)_{s \in S} = \left(\frac{k_s}{\ell} \right)_{s \in S}$. In other words,

$$\ell m_s = dk_s \text{ for all } s \in S$$

and since d and ℓ are coprime, it follows that $d \mid m_s$ for all $s \in S$. As $(m_s)_{s \in S}$ is a unimodular vector, it follows that $d = 1$, $\ell m_s = k_s$ for all $s \in S$ and $\tilde{e} = \ell e$. \square

Definition 6.8. Let (R, M) be as in Convention 2.1 and let $S \subseteq R$ be a balanced set, $n = |S|$ and A the partition matrix of the partition associated to S (Definition 5.5). We define the *equalizing polynomial* of S as

$$f = \prod_{s \in S} (x - s)^{m_s}$$

where $(m_s)_{s \in S} \in \mathbb{N}^n$ is the uniquely determined unimodular solution over the positive integers to $A\mathbf{x} = e\mathbf{1}_n$ with $e \in \mathbb{N}_0$.

Remark 6.9. In the special case where S is a complete set of residues of M^k , all roots of the equalizing polynomials are simple roots. The vector of multiplicities $\mathbf{1}_n$ is a solution of $A\mathbf{x} = e\mathbf{1}_n$ for some $e \in \mathbb{N}$ because every row contains the same elements in a different order. It is therefore the unique unimodular solution.

Lemma 6.10. *Let (R, M) be as in Convention 2.1, $S \subseteq R$ a balanced subset, f the equalizing polynomial of S .*

Then $\mathcal{R}(f) = \mathcal{P}(f)$.

Proof. This follows from the definition of the equalizing polynomial and Lemma 5.7. \square

By means of the equalizing polynomial, we are now ready to show that every balanced set occurs as the root set of a split absolutely irreducible polynomial.

Theorem 1. *Let R be a discrete valuation domain with finite residue field, $S \subseteq R$ a balanced subset, f the equalizing polynomial of S and d_f a generator of the fixed divisor of f .*

Then the essentially unique image-primitive polynomial $F \in \text{Int}(R)$ associated in $K[x]$ to f , namely $F = d_f^{-1} f$, is absolutely irreducible.

Proof. $\mathcal{R}(f) = \mathcal{P}(f)$ holds by Lemma 6.10 and $\sigma(\mathcal{P}(f)) = \frac{1}{q}$ by Lemma 5.3.

Let $n = |S|$. First assume that $n = 1$. Then $f = x - s$ and hence absolutely irreducible.

From now on, assume $n > 1$. We use Proposition 3.5 to show that F is absolutely irreducible. Let $g \in K[x]$ be monic with $g \neq 1$ satisfying $Q(g) \subseteq Q(f)$. This is equivalent to $g = \prod_{t \in T} (x - t)^{k_t}$ for $\emptyset \neq T \subseteq S$ and $k_t \in \mathbb{N}$ for $t \in T$ (cf. Remark 3.7). Also assume that $\mathcal{P}(g) \subseteq \mathcal{P}(f)$; we show that g is a power of f .

Since $\mathcal{P}(g) \subseteq \mathcal{P}(f)$ and $\sigma(\mathcal{P}(g)) \geq \frac{1}{q} = \sigma(\mathcal{P}(f))$ holds by Lemma 5.3, it follows that $\mathcal{P}(g) = \mathcal{P}(f)$ (see Remark 5.2(2)). Using Lemma 5.3, we conclude that

$$\mathcal{R}(g) = \mathcal{P}(g) = \mathcal{P}(f) = \mathcal{R}(f).$$

On one hand, this implies that $S = T$ since no proper subset of balanced set can have the same rich set, see Remark 5.4(2).

On the other hand, by Lemma 5.7, $\mathcal{R}(g) = \mathcal{P}(g)$ implies that $(k_s)_{s \in S}$ is a solution to $A\mathbf{x} = e\mathbf{1}_n$ for some positive integer $e \in \mathbb{N}$ where A denotes the partition matrix

of the partition associated to S . By Proposition 6.7 it follows that $k_s = \ell m_s$ with $\ell \in \mathbb{N}$ for all $s \in S$, that is, $g = f^\ell$. \square

Theorem 1 also yields sufficient conditions for absolute irreducibility in the global case.

Corollary 6.11. *Let D be a Dedekind domain, $f = \prod_{s \in S} (x - s)^{m_s}$ for a finite subset S of D and $m_s \in \mathbb{N}$ for $s \in S$ such that the fixed divisor of f is a principal ideal, generated by $d_f \in D$.*

If there exists a prime ideal P of D with finite residue field such that S is (D_P, P_P) -balanced and f is the equalizing polynomial of S , then $d_f^{-1} f$ is absolutely irreducible in $\text{Int}(D)$.

Proof. $\text{Int}(D)$ behaves well under localization, that is, $\text{Int}(D)_P = \text{Int}(D_P)$ holds for every Noetherian domain D , cf. [4, Theorem I.2.3]. Therefore $F = d_f^{-1} f \in \text{Int}(D)$ is an element of $\text{Int}(D_P)$ and absolutely irreducible in $\text{Int}(D_P)$ by Theorem 1.

Now, let $m \in \mathbb{N}$ and assume that $F^m = G_1 G_2$ for non-constant polynomials G_1 and G_2 in $\text{Int}(D)$. It follows from the absolute irreducibility of f in $\text{Int}(D_P)$ that, for $i = 1, 2$, there exist integers $e_i \geq 1$ with $e_1 + e_2 = m$ and non-zero elements u_i in the quotient field of D with $u_1 u_2 = 1$ such that

$$G_i = u_i F^{e_i}.$$

Now, since F^{e_i} is image-primitive and $G_i \in \text{Int}(D)$, it follows that $u_i \in D$ for $i = 1, 2$. Since $u_1 u_2 = 1$, G_i is associated to F^{e_i} in $\text{Int}(D)$ for $i = 1, 2$. The assertion follows. \square

7. CHARACTERIZATION OF SPLIT ABSOLUTELY IRREDUCIBLE POLYNOMIALS

We now give a completely general characterization of absolutely irreducible polynomials in $\text{Int}(R)$ which split over K . First, we cover those whose roots are in R ; and, finally, all split absolutely irreducible integer-valued polynomials are characterized in Corollary 7.2.

Theorem 2. *Let R be a discrete valuation domain with finite residue field and K its quotient field. Let*

$$f = \prod_{s \in S} (x - s)^{m_s} \quad \text{and} \quad F = c^{-1} f,$$

where $\emptyset \neq S \subseteq R$ is a finite set and for each $s \in S$, $m_s \in \mathbb{N}$ a positive integer and $c \in K \setminus \{0\}$.

Then F is absolutely irreducible in $\text{Int}(R)$ if and only if

- (1) S is balanced.
- (2) f is the equalizing polynomial of S .
- (3) c is a generator of the fixed divisor of f .

Proof. The equivalence is trivially true when $|S| = 1$. Now assume $|S| > 1$.

If S is a balanced set, $f = f_S$ its equalizing polynomial and c a generator of the fixed divisor of f_S , then F is absolutely irreducible by Theorem 1.

Conversely, assume that F is absolutely irreducible. Let \mathcal{C}_S be the partition associated to S . Let $S' \subseteq S$ be a balanced set associated to S , cf. Definition 4.7. Then S' is a balanced set such that $\mathcal{C}_{S'} = \mathcal{C}_S$ and $\mathcal{R}(S') \subseteq \mathcal{R}(S)$, cf. Remark 4.6(4).

First, we show that $S = S'$. Let g be the equalizing polynomial of S' . Then, using Lemmas 6.10 and 4.8 we conclude that

$$\mathcal{P}(g) = \mathcal{R}(g) = \mathcal{R}(S') \subseteq \mathcal{R}(S) = \mathcal{R}(f) \subseteq \mathcal{P}(f) \quad (13)$$

holds. Moreover, $Q(g) \subseteq Q(f)$ since $S' \subseteq S$. By Corollary 3.6, F being absolutely irreducible implies that $Q(g) = Q(f)$ and $\mathcal{P}(g) = \mathcal{P}(f)$.

Therefore $S = S'$ is a balanced set and $g = f_S$ its equalizing polynomial. It follows from Theorem 1 that $G = d_g^{-1}g$ is absolutely irreducible and f is a polynomial with $Q(f) = Q(g)$ and $\mathcal{P}(f) = \mathcal{P}(g)$. We conclude by Proposition 3.5 that $f = g^n$ and, via Lemma 3.4, $F \approx G^n$ for some $n \in \mathbb{N}$. Since F is absolutely irreducible it follows that $n = 1$ and $f = g$ is the equalizing polynomial of S and c a generator of its fixed divisor. \square

The combination of Theorem 2 and Theorem 1 establishes a bijection between those absolutely irreducible polynomials in $\text{Int}(R)$ that split over R and balanced subsets of R .

Corollary 7.1. *Let (R, M) be as in Convention 2.1. We identify polynomials in $\text{Int}(R)$ that differ only by multiplication by units of R .*

The absolutely irreducible polynomials of $\text{Int}(R)$ of the form

$$f = c^{-1} \prod_{s \in S} (x - s)^{m_s}$$

with $\emptyset \neq S \subseteq R$, each m_s a positive integer, and $c \in K \setminus \{0\}$ correspond bijectively to the balanced sets $S \subseteq R$.

The bijective correspondence is as follows: given an absolutely irreducible polynomial f , map f to its set of roots S . Conversely, given a balanced finite set $S \subseteq R$, let f be its equalizing polynomial and $d_f \in R$ a generator of the fixed divisor of f , and map S to $F = d_f^{-1}f$, that is, to the essentially unique image-primitive polynomial associated in $K[x]$ to the equalizing polynomial of S .

Corollary 7.2. *Let (R, M) be as in Convention 2.1. The absolutely irreducible polynomials in $\text{Int}(R)$ that split over K are*

- (1) *those whose roots are in R as described in Corollary 7.1 and*
- (2) *linear polynomials $ax - b$ with $a \in M$ and $b \in R \setminus M$.*

Proof. Every $F \in \text{Int}(R)$ is of the form $F = \frac{cf}{d}$ where $f \in R[x]$ is a primitive polynomial and $c, d \in R$ with $\gcd(c, d) = 1$. Moreover, if F splits over K , then f is a product of linear factors of the form $ax - b$ for coprime elements a and b in R . Therefore, exactly one of a and b is in the maximal ideal M while the other is a unit of R . If a is a unit, then $a^{-1}b \in R$. If, however, $a \in M$, then b is a unit and $ar - b$ is a unit for all $r \in R$. So, $ax - b$ by itself is absolutely irreducible and cannot be a factor of an irreducible integer-valued polynomial of degree strictly greater than 1.

It follows that the only split absolutely irreducible polynomials in $\text{Int}(R)$ that are not covered by the bijection in Corollary 7.1 are the linear polynomials of the form $ax - b$ where $a \in M$ and $b \in R \setminus M$. \square

Corollary 7.3. *Let (R, M) be as in Convention 2.1 with $|R/M| = q$ and $F \in \text{Int}(R)$ be a split polynomial with root set $S \subseteq R$.*

If F is absolutely irreducible, then $|S| \equiv 1 \pmod{q-1}$.

Proof. Note that the cardinality of a balanced set equals the number of blocks of the associated M -adic partition. It is apparent from the construction of M -adic partitions in Lemma 4.4 that we can obtain any M -adic partition by starting with the single block R and then repeatedly replacing a block that is a residue class of M^n by the q residue classes of M^{n+1} contained in it, which increases the number of blocks by $q - 1$. \square

8. APPLICATION TO GENERALIZED BINOMIAL POLYNOMIALS

In this section we discuss the absolute irreducibility of the integer-valued polynomials whose root sets are initial sequences of P -orderings. Such sequences were

already considered by Pólya [18] and Ostrowski [17] in their investigation of regular bases of rings of integer-valued polynomials on rings of integers in number fields. In the literature, they are also known as *very well distributed and very well ordered* sequences, see [4, Ch. 2]. We follow here the terminology introduced by Bhargava [2, Section 2].

Definition 8.1. Let D be a Dedekind domain and P a maximal ideal of D and v_P the discrete valuation associated to P . A P -ordering of D is a sequence $(a_i)_{i \geq 0}$ in D which satisfies

$$v_P \left(\prod_{i=0}^{k-1} (a_k - a_i) \right) \leq v_P \left(\prod_{i=0}^{k-1} (a - a_i) \right)$$

for all $k \geq 0$ and all $a \in D$.

Remark 8.2. The sequence $(i)_{i \geq 1}$ of consecutive natural numbers is a $p\mathbb{Z}$ -ordering of the ring \mathbb{Z} of integers for each prime number p .

Fact 8.3 (Bhargava [2, Theorem 1, Lemma 3]). *The sequence of P -adic valuations*

$$\alpha_P(k) := v_P \left(\prod_{i=0}^{k-1} (a_k - a_i) \right)$$

does not depend on the choice of the P -ordering; it is intrinsic to D . Moreover, for each $k \geq 0$, there are only finitely many prime ideals for which $\alpha_P(k) \neq 0$.

Definition 8.4. Let D be a Dedekind domain and $k \in \mathbb{N}_0$. The *generalized factorial* of k with respect to D is defined as the ideal

$$\text{fac}_D(k) = \prod_{P \in \max(D)} P^{\alpha_P(k)}$$

where $\max(D)$ denotes set of maximal ideals of D .

Remark 8.5. For $n \in \mathbb{N}$, the generalized factorial $\text{fac}_{\mathbb{Z}}(n) = n!\mathbb{Z}$ is the ideal generated by the usual factorial $n!$.

Remark 8.6. Let D be a Dedekind domain and P a maximal ideal of D whose residue field is of finite order q and let $(a_i)_{i \geq 0}$ be a P -ordering and $n \in \mathbb{N}$.

- (1) Every choice of q^n consecutive elements of $(a_i)_{i \geq 0}$ is a complete system of residues modulo P^n . Every complete system of residues modulo P^n is the initial sequence of a P -ordering.
- (2) The sequence $(a_i)_{i \geq 0}$ is a PD_P -ordering of the localization D_P of D at P and a $P\widehat{D}$ -ordering of the P -adic completion \widehat{D} of D .

Remark 8.7. Note that for any P -ordering $(a_i)_{i \geq 0}$

$$v_P \left(d \left(\prod_{i=0}^m (x - a_i) \right) \right) = \alpha_P(m)$$

holds. This can be seen by the following argument. Any choice of m consecutive elements of $(a_i)_{i \geq 0}$, for simplicity say a_i with $0 \leq i \leq m-1$, contains exactly $\lfloor \frac{m}{q^j} \rfloor$ complete systems of residues modulo q^j for all $j \geq 1$ (where $\lfloor \cdot \rfloor$ denotes the floor operator on \mathbb{Q}).

Note that a_m is congruent to exactly $\left\lfloor \frac{m}{q^j} \right\rfloor$ elements modulo P^j for $j \geq 0$. In addition, $\max_{0 \leq i \leq m-1} v(a_m - a_i) = n$, where $n \in \mathbb{N}$ with $q^n \leq m < q^{n+1}$. Therefore,

$$\begin{aligned} \alpha_P(m) &= v_P \left(\prod_{i=0}^{m-1} (a_m - a_i) \right) = \sum_{j=1}^n |\{1 \leq i \leq m-1 \mid v(a_m - a_i) \geq j\}| \\ &= \sum_{j \geq 1} \left\lfloor \frac{m}{q^j} \right\rfloor = \sum_{j=0}^{n-1} q^j = \alpha_P(q^n). \end{aligned}$$

Definition 8.8. Let D be a Dedekind domain and assume that $k \geq 1$ is an integer such that $\text{fac}_D(k) = cD$ is a principal ideal and \mathcal{P} be the (finite) set of prime ideals which contain c . Further, let $(a_i)_{i=0}^{k-1}$ be a sequence in D such that, for each $P \in \mathcal{P}$, $(a_i)_{i=0}^{k-1}$ is an initial sequence for a P -ordering. We call

$$c^{-1} \prod_{i=0}^{k-1} (x - a_i)$$

a *generalized binomial polynomial of degree k over D* .

We now turn our attention to generalized binomial polynomials of degree q^n where q is the (finite) order of a maximal ideal P of a Dedekind domain and $n \in \mathbb{N}$. Let S be a choice of q^n consecutive elements of a P -ordering $(a_i)_{i \geq 0}$ of D . It follows from Remark 8.6 that S is a system of representatives of the residue classes of P^n . Therefore, considered as subset of the discrete valuation domain $R = D_P$ with maximal ideal $M = PD_P$, S is a (D_P, P_P) -balanced set. By Remark 6.9, its equalizing polynomial is $\prod_{s \in S} (x - s)$. (Note that a segment of consecutive elements of a P -ordering whose length is not a power of q is not a balanced set.)

In view of this discussion, the following assertion now follows from Corollary 6.11.

Corollary 8.9. *Let D be a Dedekind domain and P be a prime ideal of D with finite index q and let $n \in \mathbb{N}$ such that the generalized factorial $\text{fac}_D(q^n) = (c)$ is a principal ideal of D .*

Then, for every P -ordering $(a_i)_{i \geq 0}$, the polynomial

$$c^{-1} \prod_{i=0}^{q^n - 1} (x - a_i)$$

is absolutely irreducible in $\text{Int}(D)$.

Remark 8.10. Let p be an integer prime number and $n \in \mathbb{N}$.

- (1) It follows by Corollary 8.9 that the (classical) binomial polynomial $\binom{x}{p^n}$ is absolutely irreducible in $\text{Int}(\mathbb{Z})$. Note that this is a special case of a result of Rissner and Windisch [20, Theorem 1], who have shown that $\binom{x}{m}$ is absolutely irreducible in $\text{Int}(\mathbb{Z})$ for all $m \in \mathbb{N}$.
- (2) A new result which immediately follows from by Corollary 8.9 is that $\binom{x}{p^n}$ is absolutely irreducible in $\text{Int}(\mathbb{Z}_{(p)})$ as well as in $\text{Int}(\mathbb{Z}_p)$, where $\mathbb{Z}_{(p)}$ denotes the localization of \mathbb{Z} at p and \mathbb{Z}_p the p -adic integers.

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(Sophie Frisch) INSTITUTE OF ANALYSIS AND NUMBER THEORY, KOPERNIKUSGASSE 24/II, 8010 GRAZ, AUSTRIA

Email address: `frisch@math.tugraz.at`

(Sarah Nakato) INSTITUTE OF ANALYSIS AND NUMBER THEORY, KOPERNIKUSGASSE 24/II, 8010 GRAZ, AUSTRIA

Email address: `snakato@tugraz.at`

(Roswitha Rissner) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KLAGENFURT, UNIVERSITÄTSSTRASSE 65-67, 9020 KLAGENFURT AM WÖRTHERSEE, AUSTRIA

Email address: `roswitha.rissner@aau.at`