PRIME IDEALS IN INFINITE PRODUCTS OF COMMUTATIVE RINGS

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ABSTRACT. In this work, we present descriptions of prime ideals and in particular of maximal ideals in products $R = \prod D_\lambda$ of families $(D_\lambda)_{\lambda \in \Lambda}$ of commutative rings. We show that every maximal ideal is induced by an ultrafilter on the Boolean algebra $\prod \mathcal{P}(\max(D_\lambda))$. If every D_λ is in a certain class of rings including finite character domains and one-dimensional domains, then this leads to a characterization of the maximal ideals of R. If every D_λ is a Prüfer domain, all prime ideals of R are described. Moreover, we give an example of a (optionally non-local or local) Prüfer domain such that every non-zero prime ideal is of infinite height.

1. INTRODUCTION & PRELIMINARIES

The study of prime ideals is a central topic in commutative ring theory. It started more than a century ago when people noticed that in rings of integers in algebraic number fields elements do not necessarily factor uniquely into products of prime elements. Kummer's idea of introducing - what he called - "ideal numbers" which should again factor uniquely was formalized by Dedekind: He was the first to give the definition of an ideal. The surprising fact that ideals in rings of integers indeed factor uniquely into prime ideals was a starting point of modern algebraic number theory and ring theory.

Further problems from algebraic geometry and algebraic number theory were the motivation for investigating commutative rings in a more general framework. Since then, prime ideals have turned out to be very useful, because they reflect many algebraic and arithmetical properties of a ring. For instance, in broad settings prime ideals are linked to the theory of divisors and to valuation theory in commutative rings.

From a categorical point of view, the product is one of the most natural constructions within the class of commutative rings. Moreover, it is a practical source of examples and counterexamples and it contains subrings that are themselves central objects of interest, such as rings of polynomials, rings of integer-valued polynomials and rings of continuous functions. For further reference, see the text books by Cahen and Chabert [1] on integer-valued polynomials and by Gillman and Jerison [14] on rings of continuous functions.

Prime ideals in products of commutative rings have been studied over the last thirty years.

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While the case of products over finite index sets is an easy exercise, it is still unclear in general how prime ideals in infinite products look like. To make things precise, let Λ be a set and $(D_\lambda)_{\lambda\in\Lambda}$ a family of commutative rings. Throughout this work, we denote by $R = \prod_{\lambda\in\Lambda} D_\lambda$ the product of the rings D_λ and by $\mathcal{B} = \prod_{\lambda\in\Lambda} \mathcal{P}(\max(D_\lambda))$ the product of the Boolean algebras $(\mathcal{P}(\max(D_\lambda)), \cap, \cup)$, where $\mathcal{P}(M)$ denotes the power set of a set M and $\max(D)$ is the set of all maximal ideals of a commutative ring D. Clearly, $\mathcal{B}$ is a Boolean algebra with least element $0_\mathcal{B} = (\emptyset)_{\lambda\in\Lambda}$. We denote elements $a \in R$ by $a = (a_\lambda) = (a_\lambda)_{\lambda\in\Lambda}$ and elements $Y \in \mathcal{B}$ by $Y = (Y_\lambda) = (Y_\lambda)_{\lambda\in\Lambda}$.
In 1991, Levy, Loustaunau and Shapiro [20] showed that if every D_λ is the ring of integers $\mathbb{Z}$, then the maximal ideals of R correspond to ultrafilters on $\mathcal{B}$. Moreover, in this situation, they gave a description of all prime ideals of R and investigated the order structure of chains inside $\operatorname{spec}(R)$. O'Donnell [25] generalized some of these results to maximal ideals in products of commutative rings and characterized certain classes of prime ideals in products of Dedekind domains. These considerations have been carried on by Olberding, Saydam and Shapiro in [26], [27] and [28], where prime ideals in ultraproducts of commutative rings are explored in very broad settings.
Our aim is to extend the initial approach by Levy, Loustaunau and Shapiro to more general situations, and thereby recover and strengthen many results mentioned above. To give one example, Theorem 3.3 of our work describes all prime ideals of arbitrary products of Prüfer domains. The concept of a Prüfer domain is a common generalization of those of a Dedekind domain (including rings of integers in algebraic number fields) to the non-Noetherian setting and of a valuation domain to the non-local setting. Moreover, Prüfer domains are important objects in commutative ring theory and have been studied intensively. For a general introduction, see [15, Chapter IV]. The book by Fontana, Huckaba and Papick [10] contains collected topics on Prüfer domains.

Ultrafilters on Boolean algebras. For an introduction to Boolean algebras, see [19]. We tread ultrafilters in two different ways, that nevertheless can be summarized under one concept:

(1) Let $(B, \wedge, \vee)$ be a Boolean algebra. We denote by 0 the minimal element of B, by $\neg$ the complement operation on B and by $\leq$ the canonical order relation on B. A non-empty subset $\mathcal{U}$ of B is called a *filter in* B if it satisfies the following conditions:
   (i) $0 \notin \mathcal{U}$.
   (ii) For all $X, Y \in \mathcal{U}$, it follows that $X \wedge Y \in \mathcal{U}$.
   (iii) For all $Y \in \mathcal{U}$ and all $Z \in B$, we have that $Y \leq Z$ implies $Z \in \mathcal{U}$.
   A filter $\mathcal{U}$ in B is called an *ultrafilter in* B if it satisfies in addition
   (iv) For all $Y \in B$, we have that either $Y \in \mathcal{U}$ or $\neg Y \in \mathcal{U}$.

(2) If $(B, \wedge, \vee) = (\mathcal{P}(\Lambda), \cap, \cup)$, then we have $0 = \emptyset$ and $\neg A = \Lambda \setminus A$ for every $A \subseteq \Lambda$, and $\leq$ equals set-theoretic inclusion. Moreover, we call an ultrafilter $\mathcal{U}$ in $\mathcal{P}(\Lambda)$ an *ultrafilter on* Λ (as it is usual) and the above properties translate as follows:
   (i) $\emptyset \notin \mathcal{U}$.
   (ii) For all $A, B \in \mathcal{U}$, it follows that $A \cap B \in \mathcal{U}$.
   (iii) For all $B \in \mathcal{U}$ and all $C \subseteq \Lambda$, we have that $B \subseteq C$ implies $C \in \mathcal{U}$.
   (iv) For all $A \subseteq \Lambda$, we have that either $A \in \mathcal{U}$ or $\Lambda \setminus A \in \mathcal{U}$.

(3) If $B = \mathcal{B} = \prod_{\lambda\in\Lambda} \mathcal{P}(\max(D_\lambda))$, then for all $Y, Z \in \mathcal{B}$, we have that $Y \wedge Z = (Y_\lambda \cap Z_\lambda)_{\lambda\in\Lambda}$, $Y \vee Z = (Y_\lambda \cup Z_\lambda)_{\lambda\in\Lambda}$, $\neg Y = (\max(D_\lambda) \setminus Y_\lambda)_{\lambda\in\Lambda}$ and $0 = 0_\mathcal{B} = (\emptyset)_{\lambda\in\Lambda}$. Furthermore, we have $Y \leq Z$ if and only if $Y_\lambda \subseteq Z_\lambda$ for all $\lambda \in \Lambda$.

(4) A non-empty subset $M \subseteq B$ is said to have the *finite intersection property*, if for all $Y_1, \ldots, Y_n \in M$ we have that $Y_1 \wedge \ldots \wedge Y_n \neq 0$. If $M \subseteq B$ has the finite intersection

property, then $\mathcal{F} = \{F \in B \mid \exists Y_1, \ldots, Y_n \in M \; Y_1 \wedge \ldots \wedge Y_n \leq F\}$ can easily be seen to be a filter in B containing M. Moreover, it holds that every filter in B is contained in some ultrafilter in B. This follows from the fact that ultrafilters in B are exactly the maximal elements with respect to set-theoretical inclusion in the set of all filters on B.

(5) It is not hard to see that if $\mathcal{U}$ is an ultrafilter in B, then for all $X, Y \in B$, if $X \vee Y \in \mathcal{U}$, then $X \in \mathcal{U}$ or $Y \in \mathcal{U}$.

The above facts will be used throughout this work without any additional reference.

The Skolem-property. A subring T of $R = \prod D_\lambda$ is said to have the *Skolem-property* if for all $a^{(1)}, \ldots, a^{(n)} \in T$ such that the ideal $(a^{(1)}_\lambda, \ldots, a^{(n)}_\lambda)$ is equal to D_λ for all $\lambda \in \Lambda$, it follows that $(a^{(1)}, \ldots, a^{(n)}) = T$.

The Skolem-property introduced here is a generalization of the particular case where $T = \mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\} \subseteq \prod D$, where D is a domain with quotient field K. For a deeper insight into this circle of ideas, see [2], [3], [4], [5], [13], [21] and [22].
In section 2, it is shown that T having the Skolem-property is equivalent to every maximal ideal of T being induced by an ultrafilter in $\mathcal{B}$. Note that R always has the Skolem-property. Moreover, the ultrafilters in $\mathcal{B}$ inducing maximal ideals of R can be characterized if every D_λ is in the class of commutative rings D satisfying the following property, which we call (+):
For all $r \in D$ and $a \in D \setminus \{0\}$ there exists $d \in D$ such that d is in every maximal ideal containing a but not containing r and d is in no maximal ideal containing r.
It is shown that finite character domains and one-dimensional domains satisfy (+).
We also investigate the case where every ultrafilter on $\mathcal{B}$ induces a maximal ideal of R. It turns out that this property has strong connections to the topological assumption of proconstructability on the maximal spectra of the component rings D_λ.
Further considerations in section 2 describe the minimal prime ideals of subrings $T \subseteq R$. In particular, we present a proof of the fact that every prime ideal of a product of domains R contains exactly one minimal prime ideal.

First-order sentences and ultraproducts. In section 3, we make use of some classical terms of model theory including first-order formulas and ultraproducts, which we only consider in the special case of the language of rings including $+$, $\cdot$, 0 and 1. Roughly speaking, a first order sentence in this language is a formula only using $=$, $+$, $\cdot$, 0, 1, variables and logical symbols such as quantifiers and sentential connectives, but in such a way that variables only range over the elements of the ring.
If $\mathcal{F}$ is an ultrafilter on Λ, we denote by $R^* = \prod_{\lambda \in \Lambda}^{\mathcal{F}} D_\lambda$ the ultraproduct of the D_λ, which is the ring that is constructed by identifying elements $r, s \in R$ with the property that $\{\lambda \in \Lambda \mid r_\lambda = s_\lambda\}$ is in $\mathcal{F}$.
Ultrafilters and ultraproducts are playing an increasingly important role in commutative ring theory, for instance, in the work of Olberding (cf. [26], [27], [28]), Fontana and Loper (cf. [8], [9], [12], [23, section 5], [24]), and Schoutens (cf. [29], [30]).
We will extensively use the following fundamental theorem for ultraproducts [6, Theorem 4.1.9]:

Theorem of Łoś. A first order sentence φ is satisfied by R^* if and only if the set of all $\lambda \in \Lambda$ such that D_λ satisfies φ is in $\mathcal{F}$.

Using the Theorem of Łoś, it follows in particular that, if every D_λ is an integral domain (respectively a field) with quotient field K_λ, then so is R^*, and it can be easily seen that its quotient field K^* is isomorphic to the ultraproduct of the K_λ. For a more precise and general treatment of the introduced concepts, see [6].

In section 3, we apply the fact that being a Prüfer domain is preserved by ultraproducts [26, Proposition 2.2]. Knowing this, we are able to describe the valuation on the quotient field K^* of an ultraproduct R^* of Prüfer domains D_λ having as a valuation ring the localization R^*_M at a maximal ideal $M \subseteq R^*$. By a common generalization of concepts introduced in [20] and [27], we are then able to describe all prime ideals in R when each D_λ is a Prüfer domain. This leads us to the fact that (in the same situation) every non-minimal prime ideal of R contained in a certain type of maximal ideal (that always exists) is of infinite height. Finally, we give an example of a Prüfer domain such that every non-zero prime ideal is of infinite height, which can be chosen to be either local (so a valuation domain) or non-local.

2. Maximal ideals and minimal prime ideals

Describing all maximal ideals. Let D be a commutative ring. For an ideal $I \subseteq D$, we denote by $\mathscr{V}(I)$ the set of all maximal ideals of D containing I and by $\mathscr{D}(I) = \max(D) \setminus \mathscr{V}(I)$. If $I = (a_1, \ldots, a_n)$ is finitely generated, we write $\mathscr{V}(I) = \mathscr{V}(a_1, \ldots, a_n)$.
For an element $a \in R = \prod D_\lambda$, we set $S(a) = (\mathscr{V}(a_\lambda))_{\lambda\in\Lambda} \in \mathcal{B} = \prod \mathcal{P}(\max(D_\lambda))$. Moreover, if $\mathcal{U}$ is a filter in $\mathcal{B}$ and $T \subseteq R$ is a subring, we define

$$(\mathcal{U})^T = \{a \in T \mid S(a) \in \mathcal{U}\},$$

where T is omitted whenever the context determines it.

Lemma 2.1. Let $T \subseteq R$ be a subring, $a, b \in R$ and $\mathcal{U}$ be a filter in $\mathcal{B}$. Then the following assertions hold:

(1) $S(a) \wedge S(b) = (\mathscr{V}(a_\lambda, b_\lambda))_{\lambda\in\Lambda}$.
(2) $S(a) \vee S(b) = S(ab)$.
(3) $(\mathcal{U})$ is an ideal of T.
(4) If $\mathcal{U}$ is an ultrafilter in $\mathcal{B}$, then $(\mathcal{U})$ is a prime ideal of T.

Proof. (1), (2) and (3) follow immediately from the relevant definitions.
For the proof of (4), let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ and note that $1 \notin (\mathcal{U})$, because $S(1) = 0_{\mathcal{B}} \notin \mathcal{U}$. If now $a, b \in T$ such that $ab \in (\mathcal{U})$, then by (2) we have that $S(a) \vee S(b) = S(ab) \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, it follows that $S(a) \in \mathcal{U}$ or $S(b) \in \mathcal{U}$ and therefore $a \in (\mathcal{U})$ or $b \in (\mathcal{U})$. ☐

Proposition 2.2. For a subring $T \subseteq R = \prod_{\lambda\in\Lambda} D_\lambda$ the following assertions are equivalent:

(a) T has the Skolem-property.
(b) For every proper ideal $\mathfrak{A} \subseteq T$ the set $\{S(a) \mid a \in \mathfrak{A}\} \subseteq \mathcal{B}$ satisfies the finite intersection property.
(c) Every proper ideal of T is contained in an ideal of the form $(\mathcal{U})$, where $\mathcal{U}$ is an ultrafilter in $\mathcal{B}$.
(d) Every maximal ideal of T is of the form $(\mathcal{U})$ for some ultrafilter $\mathcal{U}$ in $\mathcal{B}$.

Proof. "(a) ⇒ (b)": Let $\mathfrak{A} \subseteq T$ be a proper ideal and $a^{(1)}, \ldots, a^{(n)} \in \mathfrak{A}$. Then, by Lemma 2.1(1), it follows that $S(a^{(1)}) \wedge \ldots \wedge S(a^{(n)}) = (\mathscr{V}(a^{(1)}_\lambda, \ldots, a^{(n)}_\lambda))$. Assume to the contrary that

$\mathscr{V}(a_\lambda^{(1)},\dots,a_\lambda^{(n)})=\emptyset$ for all $\lambda\in\Lambda$. Then $(a_\lambda^{(1)},\dots,a_\lambda^{(n)})=D_\lambda$ for all $\lambda\in\Lambda$ and by the Skolem-property we have that $\mathfrak{A}\supseteq(a^{(1)},\dots,a^{(n)})=T$, which is a contradiction.
"(b) $\Rightarrow$ (c)": Let $\mathfrak{A}\subseteq T$ be a proper ideal. Then, by (b), we can pick an ultrafilter $\mathcal{U}$ in $\mathcal{B}$ such that $\{S(a)\mid a\in\mathfrak{A}\}\subseteq\mathcal{U}$. Now it follows by definition that $\mathfrak{A}\subseteq(\mathcal{U})$.
"(c) $\Rightarrow$ (d)": This is clear.
"(d) $\Rightarrow$ (a)": Let $a^{(1)},\dots,a^{(n)}\in T$ such that $\mathfrak{A}=(a^{(1)},\dots,a^{(n)})$ is a proper ideal of T. Let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ such that $\mathfrak{A}\subseteq(\mathcal{U})$. We want to show that $(a_\lambda^{(1)},\dots,a_\lambda^{(n)})$ is proper for some $\lambda\in\Lambda$. Assume to contrary that $(a_\lambda^{(1)},\dots,a_\lambda^{(n)})=D_\lambda$ for all $\lambda\in\Lambda$. Then $0_\mathcal{B}=(\mathscr{V}(a_\lambda^{(1)},\dots,a_\lambda^{(n)}))=S(a^{(1)})\wedge\dots\wedge S(a^{(n)})\in\mathcal{U}$, which is a contradiction. ☐

Corollary 2.3. *Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of commutative rings. Then every maximal ideal of $R=\prod D_\lambda$ is of the form $(\mathcal{U})$ for some ultrafilter $\mathcal{U}$ in $\mathcal{B}$.*

Characterizing ultrafilters that induce maximal ideals.

Definition 2.4. A ring D is said to satisfy *property* $(+)$ if for all $r\in D$ and for all non-zero $a\in D$, there exists $d\in D$ such that d is in every maximal ideal of D containing a but not containing r and d is in no maximal ideal of D containing r.

We will see that property $(+)$ gives us a setting, where we can characterize the ultrafilters in $\mathcal{B}$ that induce maximal ideals of R.
We first give some easy equivalences to property $(+)$, which will help us to give examples of classes of domains satisfying and not satisfying it. To do this, we need the following fact, which follows immediately from [7, Corollary 3]: If I is an ideal in a ring D and $r\in D$, then $I\subseteq\bigcup_{Q\in\mathscr{V}(r)}Q$ implies that $I\subseteq Q$ for some $Q\in\mathscr{V}(r)$.

Lemma 2.5. Let D be a ring. Then the following assertions are equivalent:

(a) D satisfies property $(+)$.
(b) For all $r\in D$, $a\in D\setminus\{0\}$ and $Q\in\mathscr{V}(r)$ we have that $(\bigcap_{M\in\mathscr{V}(a)\setminus\mathscr{V}(r)}M)\setminus Q\neq\emptyset$.
(c) For all $r\in D$, $a\in D\setminus\{0\}$ and for all maximal ideals $Q\subseteq D$ we have that $\bigcap_{M\in\mathscr{V}(a)\setminus\mathscr{V}(r)}M\subseteq Q$ implies that there exists some $M\in\mathscr{V}(a)\setminus\mathscr{V}(r)$ such that $M\subseteq Q$.
(d) For all $r\in D$, $a\in D\setminus\{0\}$ and for all maximal ideals $Q\subseteq D$ we have that $\bigcap_{M\in\mathscr{V}(a)\setminus\mathscr{V}(r)}M\subseteq Q$ implies that $Q\in\mathscr{D}(r)$.
(e) For all $r\in D$ and for all non-zero $a\in D$, there exists $d\in D$ such that the containment $\mathscr{V}(a)\cap\mathscr{D}(r)\subseteq\mathscr{V}(d)\subseteq\mathscr{D}(r)$ holds.

Proof. The equivalence of (b), (c) and (d) is clear. Also (b) follows immediately from (a). Moreover, (a) and (e) are trivially equivalent. It now suffices to prove "(b) $\Rightarrow$ (a)". So assume that (a) does not hold. Then $\bigcap_{M\in\mathscr{V}(a)\setminus\mathscr{V}(r)}M\subseteq\bigcup_{Q\in\mathscr{V}(r)}Q$ for some $r\in D$ and some non-zero $a\in D$. By the prime-avoidance-like statement before the lemma, it follows that $\bigcap_{M\in\mathscr{V}(a)\setminus\mathscr{V}(r)}M\subseteq Q$ for some $Q\in\mathscr{V}(r)$, which contradicts (b). ☐

Example 2.6. If D is a domain of finite character, i.e. every $a\in D\setminus\{0\}$ is contained in only finitely many maximal ideals of D, then it is immediate by (c) in Lemma 2.5 and the fact that every maximal ideal $Q\subseteq D$ is prime that D satisfies $(+)$.
In particular, one-dimensional Noetherian domains (and therefore also principal ideal domains) satisfy $(+)$.

In the case that D does not have finite character, the situation is much more involved, as we want to illustrate by the next example. Nevertheless, Proposition 2.8 will enlarge the class of rings of which we know that they satisfy (+) into an important direction.

Example 2.7.
(1) If K is a field and $n \geq 2$, then the polynomial ring in n indeterminates over K is a Noetherian factorial domain of Krull dimension n that is not Prüfer and does not satisfy property (+).
(2) The polynomial ring $\mathbb{Z}[x]$ is a two-dimensional Noetherian factorial domain that is not Prüfer and does not satisfy property (+).
(3) The ring of integer-valued polynomials $\operatorname{Int}(\mathbb{Z})$ is a two-dimensional non-Noetherian Prüfer domain not satisfying (+).

Proposition 2.8. Every one-dimension domain satisfies (+).

Proof. Let D be one-dimensional, $r \in D$ and $a \in D \setminus \{0\}$. Note that $Z = \bigcap_{M \in \mathscr{V}(a) \cap \mathscr{D}(r)} M$ is an intersection of prime ideals with $a \in Z$. Therefore Z is a non-zero radical ideal of D. If $Z = D$, the assertion is trivial, so assume that Z is a proper ideal, which implies that D/Z is a reduced zero-dimensional ring (i.e. von Neumann regular).
Since $(r + Z)$ is a principal ideal of D/Z, there exists $e \in D$ such that $e + Z$ is idempotent in D/Z and $(r+Z) = (e+Z)$. We define $d = 1 - e$ and claim that d is the right choice for property (+).
Let $M \in \mathscr{V}(a) \cap \mathscr{D}(r)$. Then $r \notin M$ and this implies $r + Z \notin M/Z$. (For if we had $r + Z \in M/Z$, then we could pick $m \in M$ such that $r + Z = m + Z$. But then $r - m \in Z \subseteq M$, which would imply $r \in M$, a contradiction.) It follows that $e + Z \notin M/Z$ and therefore $d + Z \in M/Z$. With the same argument as before, we get $d \in M$, so $M \in \mathscr{V}(d)$.
Now let $M \in \mathscr{V}(d)$. Then $d \in M$, so $d + Z \in M/Z$. Therefore $e + Z \notin M/Z$, which implies $r + Z \notin M/Z$ and hence $r \notin M$. It follows that $M \in \mathscr{D}(r)$. □

Note that Proposition 2.8 gives also rise to examples of domains satisfying (+) and not being of finite character. For instance, let $\bar{\mathbb{Z}}$ be the integral closure of $\mathbb{Z}$ in some algebraic closure of $\mathbb{Q}$. Then $\bar{\mathbb{Z}}$ is a one-dimensional Prüfer domain but it is not of finite character. Indeed, every prime number $p \in \mathbb{Z}$ is contained in infinitely many maximal ideals of $\bar{\mathbb{Z}}$.
We now turn back to the investigation of maximal ideals of $R = \prod D_\lambda$ and ultrafilters in $\mathcal{B} = \prod \mathcal{P}(\max(D_\lambda))$.

Proposition 2.9. Let $(D_\lambda)_{\lambda \in \Lambda}$ be a family of rings satisfying (+) and let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ containing an element of the form $(\mathscr{V}(a_\lambda))_{\lambda \in \Lambda}$, where $a_\lambda \in D_\lambda \setminus \{0\}$ for all $\lambda \in \Lambda$. Then $(\mathcal{U})$ is a maximal ideal of $R = \prod D_\lambda$.

Proof. Let $r \in R \setminus (\mathcal{U})$ and let $(a_\lambda)_{\lambda \in \Lambda}$ be a family such that $a_\lambda \in D_\lambda \setminus \{0\}$ for all $\lambda \in \Lambda$ and $(\mathscr{V}(a_\lambda))_{\lambda \in \Lambda} \in \mathcal{U}$. Since every D_λ satisfies (+), for each $\lambda \in \Lambda$ we can pick some $d_\lambda \in D_\lambda$ such that $\mathscr{V}(a_\lambda) \cap \mathscr{D}(r_\lambda) \subseteq \mathscr{V}(d_\lambda) \subseteq \mathscr{D}(r_\lambda)$ and define $d = (d_\lambda)_{\lambda \in \Lambda}$. Since $r \notin (\mathcal{U})$, it follows that $S(r) \notin \mathcal{U}$ and therefore $(\mathscr{D}(r_\lambda)) = \neg S(r) \in \mathcal{U}$, because $\mathcal{U}$ is an ultrafilter. Hence we have $S(d) \geq S(a) \wedge (\mathscr{D}(r_\lambda)) \in \mathcal{U}$, which implies that $S(d) \in \mathcal{U}$ and therefore $d \in (\mathcal{U})$. On the other hand, we have $(d_\lambda, r_\lambda) = D_\lambda$ for all $\lambda \in \Lambda$. By the Skolem-property of R it follows that $(d, r) = R$ and therefore $(\mathcal{U})$ is maximal. □

We now introduce two new kinds of ideals. The first one will also be the prototype of minimal prime ideals in subrings $T \subseteq R$. Let $\mathcal{F}$ be an ultrafilter on Λ and $T \subseteq R$ be a subring. Then for an element $x \in T$ we set $z(x) = \{\lambda \in \Lambda \mid x_\lambda = 0\}$ and we define

$$(0)_{\mathcal{F}}^T = \{x \in T \mid z(x) \in \mathcal{F}\}.$$

Moreover, for a family $M = (M_\lambda)_{\lambda\in\Lambda}$, where $M_\lambda \in \max(D_\lambda)$ for every $\lambda \in \Lambda$, we set $z_M(x) = \{\lambda \in \Lambda \mid x_\lambda \in M_\lambda\}$ for an element $x \in T$ and define

$$M_{\mathcal{F}}^T = \{x \in T \mid z_M(x) \in \mathcal{F}\}.$$

We write $(0)_{\mathcal{F}}^T = (0)_{\mathcal{F}}$ and $M_{\mathcal{F}}^T = M_{\mathcal{F}}$ if the choice of T is clear from the context.

Lemma 2.10. If $T \subseteq R$ is a subring such that there exists $c \in T$ where $c_\lambda \in D_\lambda$ is a non-zero non-unit for every $\lambda \in \Lambda$ and $\mathcal{F}$ is an ultrafilter on Λ, then $(0)_{\mathcal{F}}$ is a non-maximal ideal of T.

Proof. It can be easily seen that $(0)_{\mathcal{F}}$ is an ideal of T. Now let $c \in T$ as in the assumption of the lemma and let $M = (M_\lambda)$ be a family such that each M_λ is a maximal ideal of D_λ containing c_λ. Clearly, $M_{\mathcal{F}} \subseteq T$ is a proper ideal with $(0)_{\mathcal{F}} \subseteq M_{\mathcal{F}}$ and $c \in M_{\mathcal{F}} \setminus (0)_{\mathcal{F}}$. Therefore $(0)_{\mathcal{F}}$ is not maximal. □

Proposition 2.11. Let $T \subseteq R = \prod_{\lambda\in\Lambda} D_\lambda$ be a subring with the property that there exists $c \in T$ such that $c_\lambda \in D_\lambda$ is a non-zero non-unit for every $\lambda \in \Lambda$, where every D_λ is an integral domain. Moreover, let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ such that $(\mathcal{U}) \subseteq T$ is a maximal ideal. Then $\mathcal{U}$ contains an element of the form $(\mathscr{V}(a_\lambda))_{\lambda\in\Lambda}$, where $a_\lambda \in D_\lambda \setminus \{0\}$ for all $\lambda \in \Lambda$.

Proof. First, note that $\{z(x) \mid x \in (\mathcal{U})\}$ does not have the finite intersection property. For otherwise there would exist an ultrafilter $\mathcal{F}$ on Λ such that $(\mathcal{U}) \subseteq (0)_{\mathcal{F}}$, which would imply that $(0)_{\mathcal{F}}$ is maximal. A contradiction to Lemma 2.10.
So we can pick $x^{(1)}, \ldots, x^{(n)} \in (\mathcal{U})$ such that $z(x^{(1)}) \cap \ldots \cap z(x^{(n)}) = \emptyset$. Therefore for all $\lambda \in \Lambda$ we can choose $i_\lambda \in \{1, \ldots, n\}$ such that $x_\lambda^{(i_\lambda)} \neq 0$ and therefore $a_\lambda := c_\lambda \cdot x_\lambda^{(i_\lambda)}$ is a non-zero non-unit of D_λ. If we now set $a = (a_\lambda)_{\lambda\in\Lambda}$, then $(\mathscr{V}(a_\lambda)) = (\mathscr{V}(c_\lambda \cdot x_\lambda^{(i_\lambda)})) \geq (\mathscr{V}(c_\lambda \cdot x_\lambda^{(1)}, \ldots, c_\lambda \cdot x_\lambda^{(n)})) = S(c \cdot x^{(1)}) \wedge \ldots \wedge S(c \cdot x^{(n)}) \in \mathcal{U}$. Therefore $(\mathscr{V}(a_\lambda)) \in \mathcal{U}$, which we wanted to show. □

Corollary 2.12. *Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of domains not being fields satisfying property (+) and let $R = \prod_{\lambda\in\Lambda} D_\lambda$. Then the maximal ideals of R are exactly the ideals of the form $(\mathcal{U})$, where $\mathcal{U}$ is an ultrafilter in the Boolean algebra $\mathcal{B} = \prod_{\lambda\in\Lambda} \mathcal{P}(\max(D_\lambda))$ containing an element of the form $(\mathscr{V}(a_\lambda))_{\lambda\in\Lambda}$ such that $a_\lambda \in D_\lambda \setminus \{0\}$ for all $\lambda \in \Lambda$.*

The finite character case. If every D_λ is a domain of finite character (i.e. every non-zero element is only contained in finitely many maximal ideals) and $T \subseteq R$ is a subring such that there exists some $c \in T$ such that every $c_\lambda \in D_\lambda$ is a non-zero non-unit, then it follows immediately from Proposition 2.11 that if $(\mathcal{U}) \subseteq T$ is a maximal ideal, then the ultrafilter $\mathcal{U}$ must contain an element $Y = (Y_\lambda)$ such that every Y_λ is finite.
The next result gives us a statement analogous to Proposition 2.9 in the finite character case.

Proposition 2.13. Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of rings such that for every $\lambda \in \Lambda$ and for every $r_\lambda \in D_\lambda$ we have that r_λ is contained either in all maximal ideals of D_λ or in only finitely many of them. Let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ containing an element $Y = (Y_\lambda)_{\lambda\in\Lambda}$ such that Y_λ is finite for every $\lambda \in \Lambda$. Then $(\mathcal{U})$ is a maximal ideal of $R = \prod_{\lambda\in\Lambda} D_\lambda$.

Proof. Let $r \in R \setminus (\mathcal{U})$. Define $a = (a_\lambda)$ such that

(1) $a_\lambda \in P$ for all $P \in \mathscr{D}(r_\lambda) \cap Y_\lambda$ and
(2) $a_\lambda \notin Q$ for all $Q \in \mathscr{V}(r_\lambda)$.

If, for $\lambda \in \Lambda$, we have that $\mathscr{V}(r_\lambda)$ is finite, then this is possible by the Chinese Remainder Theorem. If $\mathscr{V}(r_\lambda) = \max(D_\lambda)$, then this works by setting $a_\lambda = 1$. By (2) and the Skolem-property of R, it follows that $(a, r) = R$.
To see that $a \in (\mathcal{U})$, note that $S(r) \notin \mathcal{U}$ and therefore $(\mathscr{D}(r_\lambda)) = \neg S(r) \in \mathcal{U}$. Therefore $S(a) \geq (\mathscr{D}(r_\lambda)) \wedge Y \in \mathcal{U}$ and hence $a \in (\mathcal{U})$. □

Corollary 2.14. *Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of domains of finite character not being fields and let $R = \prod_{\lambda\in\Lambda} D_\lambda$. Then the maximal ideals of R are exactly the ideals of the form $(\mathcal{U})$, where $\mathcal{U}$ is an ultrafilter in the Boolean algebra $B = \prod_{\lambda\in\Lambda} \mathcal{P}(\max(D_\lambda))$ containing an element $Y = (Y_\lambda)_{\lambda\in\Lambda}$ such that Y_λ is finite for all $\lambda \in \Lambda$.*

Proconstructability of the maximal spectra. We now want to investigate the connection between a certain topological property of the $\max(D_\lambda)$ called proconstructability and the situation that for every ultrafilter $\mathcal{U}$ in $\mathcal{B}$ the ideal $(\mathcal{U}) \subseteq R = \prod D_\lambda$ is maximal.
If D is a commutative ring, then the constructible topology on $\operatorname{spec}(D)$ is a topology finer than the Zariski topology on $\operatorname{spec}(D)$ making it a compact Haussdorf space and preserving certain important properties. The easiest way to describe the closed sets in the constructible topology (which are called *proconstructible*) uses the fact that it is equal to the so-called ultrafilter topology on $\operatorname{spec}(D)$: A subset $X \subseteq \operatorname{spec}(D)$ is proconstructible if and only if for each ultrafilter F on X the prime ideal $X_F = \{r \in D \mid V(r) \cap X \in F\}$ of D is in X, where $V(r) = \{P \in \operatorname{spec}(D) \mid r \in P\}$.
If we consider the subspace $X = \max(D)$ of $\operatorname{spec}(D)$, then this property translates as follows: $X = \max(D)$ is proconstructible if and only if $X_F = \{r \in D \mid \mathscr{V}(r) \in F\}$ is maximal for each ultrafilter F on $\max(D)$.

Proposition 2.15. If $(\mathcal{U})$ is a maximal ideal of $R = \prod_{\lambda\in\Lambda} D_\lambda$ for every ultrafilter $\mathcal{U}$ in $\mathcal{B}$, then $\max(D_\lambda)$ is proconstructible in $\operatorname{spec}(D_\lambda)$ for every $\lambda \in \Lambda$.

Proof. Fix $\lambda \in \Lambda$ and set $X = \max(D_\lambda)$. As noted before the proposition, it suffices to show that $X_F = \{r \in D_\lambda \mid \mathscr{V}(r) \in F\}$ is in X for every ultrafilter F on X. So let F be an ultrafilter on X. For every $r \in D_\lambda$ consider the element $Y^{(r)} \in \mathcal{B}$ defined by setting

$$Y_\mu^{(r)} = \mathscr{D}(r) \text{ if } \mu = \lambda$$
$$Y_\mu^{(r)} = \emptyset \text{ if } \mu \neq \lambda$$

for $\mu \in \Lambda$.
Now consider the subset $\mathcal{G} = \{Y^{(r)} \mid r \in D_\lambda \setminus X_F\}$ of $\mathcal{B}$. Since F is an ultrafilter on $X = \max(D_\lambda)$ and $\mathscr{V}(r) \notin F$ for every $r \in D_\lambda \setminus X_F$, it follows that for all $r_1, \ldots, r_n \in D_\lambda \setminus X_F$ we have that $\mathscr{D}(r_1) \cap \ldots \cap \mathscr{D}(r_n) \in F$. Hence $\mathcal{G}$ has the finite intersection property as a subset of the Boolean algebra $\mathcal{B}$. Let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ such that $\mathcal{G} \subseteq \mathcal{U}$.
By assumption $(\mathcal{U}) \subseteq R$ is a maximal ideal and it can easily be seen that it contains the kernel of the projection map $p : R \to D_\lambda$. Indeed, if $r \in R$ such that $r_\lambda = 0$, then $S(r) \geq Y^{(1)} \in \mathcal{U}$. It follows that $p((\mathcal{U})) \subseteq D_\lambda$ is a maximal ideal.
Now we claim that $D_\lambda \setminus X_F \subseteq D_\lambda \setminus p((\mathcal{U}))$. If we know this, it follows that $p((\mathcal{U})) \subseteq X_F$ and therefore X_F is maximal, which is what we wanted to show.
To prove the claim, assume to the contrary that there exists $\alpha \in D_\lambda \setminus X_F$ such that $\alpha = p(f)$ for some $f \in (\mathcal{U})$, i.e. $\alpha = f_\lambda$. Since $S(f)$ and $Y^{(\alpha)}$ are in $\mathcal{U}$, it follows that $0_{\mathcal{B}} = S(f) \wedge Y^{(\alpha)} \in \mathcal{U}$, which is a contradiction. □

Definition 2.16. A commutative ring D is said to satisfy *property* (++), if for all $r \in D$ there exists some $d \in D$ such that $\mathscr{D}(r) = \mathscr{V}(d)$.

Note that if a ring D satisfies (++), then it also satisfies (+). Indeed, given $r \in D$ and $a \in D \setminus \{0\}$, let $d \in D$ such that $\mathscr{D}(r) = \mathscr{V}(d)$. Then $\mathscr{V}(a) \cap \mathscr{D}(r) \subseteq \mathscr{D}(r) = \mathscr{V}(d) \subseteq \mathscr{D}(r)$. So D satisfies (+) by Lemma 2.5.
Before we will see examples of rings with property (++), we want to illustrate how we can apply it to our description of maximal ideals of the product ring R.

Lemma 2.17. If $(D_\lambda)_{\lambda\in\Lambda}$ is a family of commutative rings satisfying (++), then $(\mathcal{U})$ is a maximal ideal of $R = \prod_{\lambda\in\Lambda} D_\lambda$ for every ultrafilter $\mathcal{U}$ in $\mathcal{B}$.

Proof. Let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ and choose $r \in R \setminus (\mathcal{U})$. Using property (++), let $d \in R$ such that $\mathscr{D}(r_\lambda) = \mathscr{V}(d_\lambda)$ for every $\lambda \in \Lambda$. Then, by the Skolem-property of R, we have that $(r, d) = R$. Moreover, since $S(r) \notin \mathcal{U}$, we have that $S(d) = (\mathscr{V}(d_\lambda)) = (\mathscr{D}(r_\lambda)) = \neg S(r) \in \mathcal{U}$, hence $d \in (\mathcal{U})$. This shows that $(\mathcal{U})$ is maximal. □

For a subset $X \subseteq \operatorname{spec}(D)$, where D is a commutative ring, we denote by $Cl^{zar}(X)$ the closure of X with respect to the Zariski topology, by $Cl^{cons}(X)$ the closure of X with respect to the constructible topology and by

$$X^{sp} = \{P \in \operatorname{spec}(D) \mid P \supseteq Q \text{ for some } Q \in X\}$$

the *specialization of* X.
It is shown in [11, Lemma 1.1] that $Cl^{zar}(X) = (Cl^{cons}(X))^{sp}$ for every $X \subseteq \operatorname{spec}(D)$. From this it follow easily that $\max(D)$ is proconstructible in $\operatorname{spec}(D)$ if and only if it is closed with respect to the Zariski topology on $\operatorname{spec}(D)$.

Proposition 2.18. Let D be a commutative ring such that $\max(D)$ is proconstructible in $\operatorname{spec}(D)$. Then D satisfies property (++).

Proof. Let J denote the Jacobson radical of D. Since $\max(D)$ is proconstructible, it follows by the remarks before the proposition that $\max(D)$ is closed with respect to the Zariski topology. In this case we have that $\{P \in \operatorname{spec}(D) \mid J \subseteq P\} = \max(D)$ and therefore $D' := D/J$ is a zero-dimensional reduced ring.
Let $r \in D$. Since D' is zero-dimensional reduced, it follows that there exists some $e \in D$ such that $e + J \in D'$ is idempotent and the principal ideals $(r + J)D'$ and $(e + J)D'$ coincide. Let $d := 1 - e$. Then it can be easily seen that $\mathscr{D}(r + J) = \mathscr{D}(e + J) = \mathscr{V}(d + J)$. From this it is clear that $\mathscr{D}(r) = \mathscr{V}(d)$. □

Note that, if D is zero-dimensional, then $\max(D) = \operatorname{spec}(D)$ is proconstructible. Also, if D is a one-dimensional domain with non-zero Jacobson radical J, then $\max(D) = V(J)$ is proconstructible. Hence both zero-dimensional rings and one-dimensional domains with non-zero Jacobson radical satisfy (++).
The next result is now an immediate consequence of Proposition 2.15, Lemma 2.17 and Proposition 2.18.

Corollary 2.19. *Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of commutative rings and $R = \prod_{\lambda\in\Lambda} D_\lambda$. Then the following assertions are equivalent:*

(a) $(\mathcal{U})$ *is a maximal ideal of R for every ultrafilter $\mathcal{U}$ in the Boolean algebra $\mathcal{B} = \prod_{\lambda\in\Lambda} \mathcal{P}(\max(D_\lambda))$.*
(b) *The subspace* $\max(D_\lambda)$ *is proconstructible in* $\operatorname{spec}(D_\lambda)$ *for every* $\lambda \in \Lambda$.
(c) D_λ *satisfies property (++) for every $\lambda \in \Lambda$, i.e. for every $r \in D_\lambda$ there exists $d \in D_\lambda$ such that $\mathscr{D}(r) = \mathscr{V}(d)$.*

In the particular case where $|\Lambda| = 1$, we get the following statement:

Corollary 2.20. *Let D be a commutative ring. Then $\max(D)$ is proconstructible in spec(D) if and only if D satisfies property (++), i.e. for every $r \in D$ there exists $d \in D$ such that $\mathscr{D}(r) = \mathscr{V}(d)$.*

Minimal prime ideals. For the rest of this section, let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of integral domains and $R = \prod D_\lambda$. Recall that for an element $x \in R$ we set $z(x) = \{\lambda \in \Lambda \mid x_\lambda = 0\}$ and define $n(x) = \Lambda \setminus z(x)$. Moreover, recall the definition of the proper ideal $(0)^T_{\mathcal{F}} = \{x \in T \mid z(x) \in \mathcal{F}\}$ of a subring $T \subseteq R$ for an ultrafilter $\mathcal{F}$ on Λ.

Proposition 2.21. Let $\mathcal{F}$ be an ultrafilter on Λ and $T \subseteq R = \prod_{\lambda\in\Lambda} D_\lambda$ be a subring.

(1) The ultraproduct $R^* = \prod^{\mathcal{F}}_{\lambda\in\Lambda} D_\lambda$ is isomorphic to $R/(0)^R_{\mathcal{F}}$.
(2) $(0)^T_{\mathcal{F}}$ is a prime ideal of T.
(3) Every minimal prime ideal of T is of the form $(0)^T_{\mathcal{F}}$ for some ultrafilter F on T.

Proof. To prove (1), note that $\varphi : R \to R^*$ mapping an element $r \in R$ to its equivalence class $r^* \in R^*$ is a surjective homomorphism. Its kernel can be easily seen to coincide with $(0)^R_{\mathcal{F}}$.
Now, to prove (2), consider the map $\iota : T/(0)^T_{\mathcal{F}} \to R/(0)^R_{\mathcal{F}}$ with $\iota(x+(0)^T_{\mathcal{F}}) := x+(0)^R_{\mathcal{F}}$. It clearly is an injective homomorphism. Moreover, by (1) and the Theorem of Łoś, $R/(0)^R_{\mathcal{F}}$ is an integral domain, hence so is $T/(0)^T_{\mathcal{F}}$. It follows that $(0)^T_{\mathcal{F}}$ is a prime ideal of T.
Finally, for the proof of (3), let $P \subseteq T$ be a minimal prime ideal and let $M = \{n(x) \mid x \in T \setminus P\}$. We claim that M has the finite intersection property. Assume to the contrary that there are $x_1, \ldots, x_n \in T \setminus P$ such that $n(x_1) \cap \ldots \cap n(x_n) = \emptyset$. Then $x_1 \cdot \ldots \cdot x_n = 0 \in P$ and therefore there exists some $i \in \{1, \ldots, n\}$ such that $x_i \in P$, which is a contradiction. Let $\mathcal{F}$ be an ultrafilter on Λ such that $M \subseteq \mathcal{F}$. Clearly, $T \setminus P \subseteq T \setminus (0)^T_{\mathcal{F}}$ and therefore $(0)^T_{\mathcal{F}} \subseteq P$. By the minimality of P it follows that $P = (0)^T_{\mathcal{F}}$. ☐

In the next lemma, we have to restrict our scope to subrings $T \subseteq R$ such that for every $Z \subseteq \Lambda$ there exists some $x \in T$ such that $Z = z(x)$. Note, that there are examples of such rings T not being equal to a product of commutative rings, e.g. let T be the ring generated (in R) by all elements $x \in R$ such that $x_\lambda \in \{0, 1\}$ for all $\lambda \in \Lambda$.

Lemma 2.22. Let $T \subseteq R = \prod D_\lambda$ be a subring such that for every $Z \subseteq \Lambda$ there exists some $x \in T$ such that $Z = z(x)$. Then $(0)^T_{\mathcal{F}}$ is a minimal prime ideal of T for every ultrafilter $\mathcal{F}$ on Λ. Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are two different ultrafilters on Λ, then $(0)^T_{\mathcal{F}} \neq (0)^T_{\mathcal{G}}$.

Proof. Let $\mathcal{F}$ be an ultrafilter on Λ and $P \subseteq T$ be a prime ideal such that $P \subseteq (0)^T_{\mathcal{F}}$. Let $x \in (0)^T_{\mathcal{F}}$ and let $y \in T$ such that $z(y) = \Lambda \setminus z(x)$. Then $x \cdot y = 0 \in P$. Since P is prime, either $x \in P$ or $y \in P$. But y cannot be an element of $P \subseteq (0)^T_{\mathcal{F}}$, because otherwise $x + y \in (0)^T_{\mathcal{F}}$ and therefore $\emptyset = z(x + y) \in \mathcal{F}$, which is a contradiction. Hence it must hold that $x \in P$.
Now, let $\mathcal{G}$ be an ultrafilter on Λ different from $\mathcal{F}$. Let $Z \in \mathcal{G} \setminus \mathcal{F}$ and $x \in T$ such that $z(x) = Z$. Then $x \in (0)^T_{\mathcal{G}} \setminus (0)^T_{\mathcal{F}}$. ☐

Corollary 2.23. *Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of commutative integral domains and let $T \subseteq R = \prod_{\lambda\in\Lambda} D_\lambda$ be a subring such that for every $Z \subseteq \Lambda$ there exists some $x \in T$ such that $Z = \{\lambda \in \Lambda \mid x_\lambda = 0\}$. Then the map $\mathcal{F} \mapsto (0)^T_{\mathcal{F}}$ is a bijection between ultrafilters on Λ and minimal prime ideals of T.*

Let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ and for every $Y \in \mathcal{U}$ set $F_Y = \{\lambda \in \Lambda \mid Y_\lambda \neq \emptyset\}$. Then we can define the collection

$$\mathcal{F}_{\mathcal{U}} = \{F_Y \mid Y \in \mathcal{U}\}$$

of subsets of Λ. It can be easily seen that $\mathcal{F}_{\mathcal{U}}$ is an ultrafilter on Λ.

Proposition 2.24. Let $\mathcal{U}$ be an ultrafilter in $\mathcal{B}$ and $\mathcal{F}$ be an ultrafilter on Λ. Then the containment $(0)_{\mathcal{F}} \subseteq (\mathcal{U})$ of ideals of $R = \prod_{\lambda\in\Lambda} D_\lambda$ holds if and only if $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$.
In particular, every prime ideal of R contains a *unique* minimal prime ideal.

Proof. Assume that $(0)_{\mathcal{F}} \subseteq (\mathcal{U})$ and let $F \in \mathcal{F}$. For a subset $M \subseteq \Lambda$ we denote by χ_M the element of R for which the entry at $\lambda \in \Lambda$ is 1 if $\lambda \in M$ and is 0 if $\lambda \notin M$. If we set $M = \Lambda \setminus F$, then $\chi_M \in (0)_{\mathcal{F}} \subseteq (\mathcal{U})$. Therefore $Y := S(\chi_M) \in \mathcal{U}$, where $Y_\lambda = \emptyset$ if $\lambda \notin F$ and $Y_\lambda = \max(D_\lambda)$ if $\lambda \in F$. So $F = \{\lambda \in \Lambda \mid Y_\lambda \neq \emptyset\}$ and therefore $F \in \mathcal{F}_{\mathcal{U}}$. Whence $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{U}}$, which implies $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$, because $\mathcal{F}$ is an ultrafilter.
Conversely, let $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$ and let $r \in (0)_{\mathcal{F}}$. Then for $M = \{\lambda \in \Lambda \mid r_\lambda \neq 0\}$ we have that $\chi_M \in (0)_{\mathcal{F}}$. Therefore $\Lambda \setminus M \in \mathcal{F}$, which implies that $\Lambda \setminus M = \{\lambda \in \Lambda \mid Y_\lambda \neq \emptyset\}$ for some $Y \in \mathcal{U}$. Clearly, we have $S(\chi_M) \geq Y$, so $S(\chi_M) \in \mathcal{U}$. Consequently, $r = r\chi_M \in (\mathcal{U})$.
For the last statement, let $\mathfrak{P} \subseteq R$ be a prime ideal. Then $\mathfrak{P}$ contains a minimal prime ideal. If $\mathfrak{Q} \subseteq \mathfrak{P}$ is a minimal prime ideal, then by Proposition 2.2 there exists an ultrafilter $\mathcal{F}$ on Λ such that $\mathfrak{Q} = (0)_{\mathcal{F}}$. In the same way, if $\mathfrak{M}$ is a maximal ideal containing $\mathfrak{P}$, then by Proposition 2.21(3) we can pick some ultrafilter $\mathcal{U}$ on $\mathcal{B}$ such that $\mathfrak{M} = (\mathcal{U})$. Since $(0)_{\mathcal{F}} \subseteq \mathfrak{P} \subseteq (\mathcal{U})$, it follows by the considerations before that $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$, so $(0)_{\mathcal{F}} = (0)_{\mathcal{F}_{\mathcal{U}}}$, which therefore is the unique minimal prime ideal contained in $\mathfrak{P}$. □

Proposition 2.24 gives a better understanding of the order structure of the set of prime ideals in the product R of integral domains in the sense that $\mathrm{spec}(R)$ is a disjoint union of partially ordered sets O, where each O has a unique minimal element. This is also a starting point for our considerations in the next section.

3. Prime ideals in products of Prüfer domains

By Proposition 2.24, if we want to characterize all prime ideals of $R = \prod_{\lambda\in\Lambda} D_\lambda$ it is sufficient to describe for every ultrafilter $\mathcal{U}$ on $\mathcal{B}$ the prime ideals $\mathfrak{P} \subseteq R$ with $(0)_{\mathcal{F}_{\mathcal{U}}} \subseteq \mathfrak{P} \subseteq (\mathcal{U})$. So from now on, fix an ultrafilter $\mathcal{U}$ in the Boolean algebra $\mathcal{B} = \prod_{\lambda\in\Lambda} \mathcal{P}(\max(D_\lambda))$ and let $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$ be the corresponding ultrafilter on Λ.
Let $R^* = \prod_{\lambda\in\Lambda}^{\mathcal{F}} D_\lambda$ be the ultraproduct of the D_λ with respect to $\mathcal{F}$. We have seen in Proposition 2.21 that R^* is isomorphic to $R/(0)_{\mathcal{F}}$. Let moreover $R^*_{\mathcal{U}}$ denote the localization of the integral domain R^* at the maximal ideal $(\mathcal{U})^*$ of R^* corresponding to $(\mathcal{U})$. Then the prime ideals $\mathfrak{P} \subseteq R$ with $(0)_{\mathcal{F}} \subseteq \mathfrak{P} \subseteq (\mathcal{U})$ are in inclusion preserving one-to-one correspondence with the prime ideals of $R^*_{\mathcal{U}}$.
For $r \in R$, we denote by r^* the image of r in R^* or the image of r in $R^*_{\mathcal{U}}$, depending on in which ring we are working in at the moment.
In the following, we want to characterize all prime ideals of R, where every D_λ is a Prüfer domain. In [20] this was done for the special case where each D_λ is the ring of integers. In [27] and [28], special types and chains of prime ideals in ultraproducts of certain commutative rings are described, including for instance all prime ideals in ultraproducts of Dedekind domains. Our investigation of prime ideals in products of general Prüfer domains is new and is different from

the one in [28] in the special case of Dedekind domains. Therefore it also gives a new viewpoint in this situation.
From now on, let D_λ be a Prüfer domain for every $\lambda \in \Lambda$.
It is shown in [26, Proposition 2.2] that "Prüfer domain" is preserved by ultraproducts. Therefore R^* is a Prüfer domain and $R^*_{\mathcal{U}}$ is a valuation domain. Let K^* be the quotient field of R^* and note that it is isomorphic to the ultraproduct with respect to $\mathcal{F}$ of the quotient fields K_λ of D_λ. Moreover we extend the notation $\mathscr{V}(r_\lambda)$ and $\mathscr{D}(r_\lambda)$ to elements $k_\lambda \in K_\lambda$, namely set $\mathscr{V}(k_\lambda) = \{M \in \max(D_\lambda) \mid v_M(k_\lambda) > 0\}$, where v_M is the valuation on K_λ corresponding to M, and let $\mathscr{D}(k_\lambda) = \max(D_\lambda) \setminus \mathscr{V}(k_\lambda)$.
In the following proposition we are able to partially describe the valuation v on K^* that has $R^*_{\mathcal{U}}$ as its valuation ring.

Valuations and prime ideals.

Proposition 3.1. Let v be the valuation on K^* having $R^*_{\mathcal{U}}$ as valuation ring. Then for $a, b \in R$, the following assertions are equivalent:

(1) $v(a^*) \geq v(b^*)$.
(2) There exists $Y \in \mathcal{U}$ such that for all $\lambda \in \Lambda$ and for all $P \in Y_\lambda$ it holds that $v_P(a_\lambda) \geq v_P(b_\lambda)$.

Proof. If either a^* or b^* is equal to 0, then the statement is trivial. So let $a^* \neq 0 \neq b^*$. Since both (1) and (2) only depend on entries a_λ and b_λ of a and b for λ in an ultrafilter set of $\mathcal{F}$ (and a,b are both non-zero on such a set), we can assume without loss of generality that $a_\lambda \neq 0 \neq b_\lambda$ for all $\lambda \in \Lambda$.
Now assume that (2) holds and let $Y \in \mathcal{U}$ such that for all $\lambda \in \Lambda$ and for all $P \in Y_\lambda$ we have $v_P(a_\lambda) \geq v_P(b_\lambda)$. Assume to the contrary that $v(a^*) < v(b^*)$. Then $0 < v(\frac{b^*}{a^*})$ and therefore $\frac{b^*}{a^*} \in (\mathcal{U})^* \subseteq R^*_{\mathcal{U}}$. Since localization commutes with forming the quotient modulo some ideal, we can write $\frac{b^*}{a^*} = (\frac{b_\lambda}{a_\lambda})^*_\lambda$. We set $Z = (\mathscr{V}(\frac{b_\lambda}{a_\lambda}))_\lambda$.
Claim: $Z \in \mathcal{U}$.
If the claim holds, we know that $Y \wedge Z \in \mathcal{U}$, so in particular $Y \wedge Z \neq 0_{\mathcal{B}}$. So we can pick $\lambda \in \Lambda$ such that $Y_\lambda \cap Z_\lambda \neq \emptyset$. Let $P \in Y_\lambda \cap Z_\lambda$. Since $P \in Y_\lambda$, we have $v_P(a_\lambda) \geq v_P(b_\lambda)$. On the other hand, since $P \in Z_\lambda = \mathscr{V}(\frac{b_\lambda}{a_\lambda})$, we have $v_P(\frac{b_\lambda}{a_\lambda}) > 0$, which implies $v_P(b_\lambda) > v_P(a_\lambda)$. This is a contradiction.
To prove the claim, first note that, since $\frac{b^*}{a^*} \in (\mathcal{U})^*$, we can pick $d \in (\mathcal{U})$ and $c \in R \setminus (\mathcal{U})$ such that $\frac{b^*}{a^*} = \frac{d^*}{c^*}$. By the same argument as in the beginning of the proof, we can choose c such that $c_\lambda \neq 0$ for all $\lambda \in \Lambda$ and we are then able to write $\frac{d^*}{c^*} = (\frac{d_\lambda}{c_\lambda})^*_\lambda$. Since $\frac{b^*}{a^*} = \frac{d^*}{c^*}$, it follows that $\frac{b}{a}$ and $\frac{d}{c}$ coincide on a set of $\mathcal{F}$ and again, since our considerations only are influenced by entries for λ in some element of $\mathcal{F}$, we may assume that $\frac{b}{a} = \frac{d}{c}$. Therefore it follows that $Z = (\mathscr{V}(\frac{d_\lambda}{c_\lambda}))_\lambda = (\{M \in \max(D_\lambda) \mid v_M(\frac{d_\lambda}{c_\lambda}) > 0\})_\lambda \geq (\{M \in \max(D_\lambda) \mid d_\lambda \in M \wedge c_\lambda \notin M\})_\lambda = (\mathscr{V}(d_\lambda))_\lambda \wedge (\mathscr{D}(c_\lambda))_\lambda \in \mathcal{U}$ by the choice of c and d. So $Z \in \mathcal{U}$ and the proof for the implication from (2) to (1) is complete.
Now assume that (1) holds and assume to the contrary that for all $Y \in \mathcal{U}$ there exists some $\lambda \in \Lambda$ and some $P \in Y_\lambda$ such that $v_P(a_\lambda) < v_P(b_\lambda)$. By (1), we have that $v(\frac{a^*}{b^*}) \geq 0$ so that $\frac{a^*}{b^*} \in R^*_{\mathcal{U}}$. Similar to the proof of the other direction, we can now pick $c \in R$ and $d \in R \setminus (\mathcal{U})$ such that $d_\lambda \neq 0$ for all $\lambda \in \Lambda$ and $(\frac{c_\lambda}{d_\lambda})^*_\lambda = \frac{c^*}{d^*} = \frac{a^*}{b^*} = (\frac{a_\lambda}{b_\lambda})^*_\lambda$. As before, it is no restriction of generality to assume that $\frac{a_\lambda}{b_\lambda} = \frac{c_\lambda}{d_\lambda}$ for all $\lambda \in \Lambda$. It follows by the choice of d that $Y := (\mathscr{D}(d_\lambda))_\lambda \in \mathcal{U}$. So by

assumption, we can pick $\lambda \in \Lambda$ and $P \in Y_\lambda$ such that $v_P(a_\lambda) < v_P(b_\lambda)$. Whence $0 \leq v_P(\frac{c_\lambda}{d_\lambda}) = v_P(\frac{a_\lambda}{b_\lambda}) < 0$, which is a contradiction. □

For every $\lambda \in \Lambda$ and $P \in \max(D_\lambda)$, we denote by S_P the totally ordered submonoid of non-negative elements (including ∞) of the value group associated to the valuation v_P of P. We define $S = \prod_{\lambda\in\Lambda}\prod_{P\in\max(D_\lambda)} S_P$ to be the product of all these monoids. We write elements $g \in S$ as $g = (g_{\lambda,P})_{\lambda\in\Lambda,P\in\max(D_\lambda)}$. For $g \in S$, define

$$(\mathcal{U})^g = \{x \in R \mid \exists Y \in \mathcal{U}\ \exists n \in \mathbb{N}\ \forall\lambda \in \Lambda\ \forall P \in Y_\lambda\ v_P(x_\lambda^n) \geq g_{\lambda,P}\}.$$

It will turn out that the sets $(\mathcal{U})^g$ are prime ideals of R and that they can be used to describe all prime ideals of R contained in $(\mathcal{U})$ and containing $(0)_\mathcal{F}$.

Proposition 3.2. For any $g \in S$ with $g_{\lambda,P} > 0$ for all $\lambda \in \Lambda$ and $P \in \max(D_\lambda)$, we have that $(\mathcal{U})^g$ is a prime ideal of R contained in $(\mathcal{U})$.

Proof. Clearly $(\mathcal{U})^g$ is an ideal. To see that it is contained in $(\mathcal{U})$, let $x \in (\mathcal{U})^g$ and choose $Y \in \mathcal{U}$ and $n \in \mathbb{N}$ such that for all $\lambda \in \Lambda$ and for all $P \in Y_\lambda$ we have $v_P(x_\lambda^n) \geq g_{\lambda,P} > 0$. It follows that $S(x^n) \geq Y \in \mathcal{U}$, so $x^n \in (\mathcal{U})$, which is a prime ideal. Therefore $x \in (\mathcal{U})$.
Finally, let $a, b \in R$ such that $ab \in (\mathcal{U})^g$ and let $Y \in \mathcal{U}$ and $n \in \mathbb{N}$ such that for all $\lambda \in \Lambda$ and all $P \in Y_\lambda$ we have $v_P(a_\lambda^n b_\lambda^n) \geq g_{\lambda,P}$. Given $\lambda \in \Lambda$ and $P \in Y_\lambda$, it follows that $g_{\lambda,P} + g_{\lambda,P} \leq v_P(a_\lambda^n b_\lambda^n) + v_P(a_\lambda^n b_\lambda^n) = v_P(a_\lambda^{2n} b_\lambda^{2n}) = v_P(a_\lambda^{2n}) + v_P(b_\lambda^{2n})$. Hence $v_P(a_\lambda^{2n}) \geq g_{\lambda,P}$ or $v_P(b_\lambda^{2n}) \geq g_{\lambda,P}$. If we define $Y_a = (\{P \in \max(D_\lambda) \mid v_P(a^{2n}) \geq g_{\lambda,P}\})_\lambda$ and $Y_b = (\{P \in \max(D_\lambda) \mid v_P(b^{2n}) \geq g_{\lambda,P}\})_\lambda$, then it follows that $Y_a \vee Y_b \geq Y$, which implies that $Y_a \vee Y_b \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, it follows that $Y_a \in \mathcal{U}$ or $Y_b \in \mathcal{U}$. Say $Y_a \in \mathcal{U}$. Then there exist $Y' \in \mathcal{U}$ (namely $Y' = Y_a$) and $n' \in \mathbb{N}$ (namely $n' = 2n$) such that for all $\lambda \in \Lambda$ and for all $P \in Y'_\lambda$ it holds that $v_P(a_\lambda^{n'}) \geq g_{\lambda,P}$, which means per definition that $a \in (\mathcal{U})^g$. □

Proposition 3.3. Let $x \in (\mathcal{U})$ and $g_{\lambda,P} = v_P(x_\lambda)$ if $P \in \mathscr{V}(x_\lambda)$ and $g_{\lambda,P} = \infty$ otherwise. Then $(\mathcal{U})^g$ is the smallest prime ideal $\mathfrak{P}$ with $(0)_\mathcal{F} \subseteq \mathfrak{P} \subseteq (\mathcal{U})$ containing x.

Proof. We already know that $(\mathcal{U})^g \subseteq (\mathcal{U})$. To see that $x \in (\mathcal{U})^g$, set $Y = S(x) \in \mathcal{U}$, $n = 1$. Then for all $\lambda \in \Lambda$ and for all $P \in Y_\lambda$, we have by definition that $v_P(x_\lambda^n) = v_P(x_\lambda) = g_{\lambda,P}$. So $x \in (\mathcal{U})^g$.
Now let $(0)_\mathcal{F} \subseteq \mathfrak{P} \subseteq (\mathcal{U})$ be a prime ideal containing x. Since the prime ideals of R containing $(0)_\mathcal{F}$ and being contained in $(\mathcal{U})$ are in inclusion preserving bijection with the prime ideals of $R_\mathcal{U}^*$, it suffices to prove the inclusion $((\mathcal{U})^g)^* \subseteq \mathfrak{P}^*$ of the corresponding prime ideals in $R_\mathcal{U}^*$. So let $r \in (\mathcal{U})^g$. We show that $r^* \in \mathfrak{P}^*$. Let $Y \in \mathcal{U}$ and $n \in \mathbb{N}$ such that for all $\lambda \in \Lambda$ and $P \in Y_\lambda$ we have $v_P(r_\lambda^n) \geq g_{\lambda,P}$ and without loss of generality choose $Y \leq S(x)$, so that for all $\lambda \in \Lambda$ and $P \in Y_\lambda$ we have $v_P(r_\lambda^n) \geq g_{\lambda,P} = v_P(x_\lambda)$ (this is possible, since $S(x) \in \mathcal{U}$ and therefore $Y \wedge S(x) \leq S(x)$ is in $\mathcal{U}$). By Proposition 3.1, it follows that $v((r^*)^n) \geq v(x^*)$, where v is the valuation on K^* having $R_\mathcal{U}^*$ as valuation ring. Therefore, x^* divides $(r^*)^n$ in $R_\mathcal{U}^*$, which implies that $(r^*)^n \in \mathfrak{P}^*$, which is a prime ideal and therefore contains r^*. This is what we wanted to show. □

Theorem 3.4. *Let $R = \prod_{\lambda\in\Lambda} D_\lambda$ where every D_λ is a Prüfer domain. The prime ideals of R are exactly the unions of prime ideals of the form $(\mathcal{U})^g$ with $g \in S$ such that $g_{\lambda,P} > 0$ for all $\lambda \in \Lambda$ and $P \in \max(D_\lambda)$.*

Proof. Since $R_\mathcal{U}^*$ is a valuation domain, the prime ideals of R contained in $(\mathcal{U})$ form a chain. Therefore every union of $(\mathcal{U})^g$ is a union of a chain of prime ideals, hence it is prime.

Conversely, let $(0)_{\mathcal{F}} \subseteq \mathfrak{P} \subseteq (\mathcal{U})$ be a prime ideal of R. For $x \in \mathfrak{P}$ we define an element $g(x) \in S$ such that for all $\lambda \in \Lambda$ and for all $P \in \max(D_\lambda)$ we have $g(x)_{\lambda,P} > 0$. Namely, set $g(x)_{\lambda,P} = v_P(x_\lambda)$ if $P \in V(x_\lambda)$ and $g(x)_{\lambda,P} = \infty$ otherwise. We claim that

$$\mathfrak{P} = \bigcup_{x \in \mathfrak{P}} (\mathcal{U})^{g(x)}.$$

By Proposition 3.3, $(\mathcal{U})^{g(x)}$ is the smallest prime ideal contained in $(\mathcal{U})$ and containing x. So $\bigcup_{x\in\mathfrak{P}} (\mathcal{U})^{g(x)} \subseteq \mathfrak{P}$. On the other hand, if $y \in \mathfrak{P}$, then by Proposition 3.3 we have that $y \in (\mathcal{U})^{g(y)}$ and therefore $y \in \bigcup_{x\in\mathfrak{P}} (\mathcal{U})^{g(x)}$. ☐

Heights of prime ideals. Recall that for every $\lambda \in \Lambda$ and $P \in \max(D_\lambda)$, we denote by S_P the totally ordered submonoid of non-negative elements (including ∞) of the value group associated to the valuation v_P of P and we defined $S = \prod_{\lambda\in\Lambda} \prod_{P \in \max D_\lambda} S_P$ to be the product of all these monoids.
We now define a relation $\ll$ on S, where

$$g \ll h :\Leftrightarrow \forall Y \in \mathcal{U}\ \forall n \in \mathbb{N}\ \exists \lambda \in \Lambda\ \exists P \in Y_\lambda \quad n \cdot g_{\lambda,P} < h_{\lambda,P}$$

for $g, h \in S$.

Lemma 3.5. Let $g, h \in S$.

(1) If $(\mathcal{U})^h \subsetneqq (\mathcal{U})^g$, then $g \ll h$.
(2) Let it hold in addition that there is some $Y \in \mathcal{U}$ and some $N \in \mathbb{N}$ such that for all $\lambda \in \Lambda$ we have $|Y_\lambda| \leq N$ (e.g. let every D_λ be semilocal with a uniform bound on the cardinality of $\max(D_\lambda)$). Then $g \ll h$ implies $(\mathcal{U})^h \subsetneqq (\mathcal{U})^g$.

Proof. To see (1), let $x \in (\mathcal{U})^g \setminus (\mathcal{U})^h$. Then there exists some $Y \in \mathcal{U}$ and some $n \in \mathbb{N}$ such that for all $\lambda \in \Lambda$ and $P \in Y_\lambda$ we have $v_P(x_\lambda^n) \geq g_{\lambda,P}$. On the other hand, for all $Y' \in \mathcal{U}$ and $n' \in \mathbb{N}$ there exists some $\lambda \in \Lambda$ and some $P \in Y'_\lambda$ such that $v_P(x_\lambda^{n'}) < h_{\lambda,P}$. It follows immediately that $g \ll h$.
By the additional assumption in statement (2), we can find $Y \in \mathcal{U}$ and $Y_1, \ldots, Y_N \in \mathcal{B}$ such that $Y = Y_1 \vee \ldots \vee Y_N$ and for each $i \in \{1, \ldots, N\}$ and $\lambda \in \Lambda$ we have $|(Y_i)_\lambda| = 1$. Moreover, since $Y \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter there exists some $i \in \{1, \ldots, N\}$ such that $Y' := Y_i \in \mathcal{U}$. For each $\lambda \in \Lambda$, let P_λ be the unique maximal ideal of D_λ contained in Y'_λ and let $x_\lambda \in D_\lambda$ such that $v_{P_\lambda}(x_\lambda) = g_{\lambda,P}$. Clearly, $(x_\lambda)_{\lambda\in\Lambda} \in (\mathcal{U})^g \setminus (\mathcal{U})^h$. ☐

We now introduce a special type of ultrafilter that will be helpful to force certain prime ideals in R to have infinite height. Let B be a Boolean algebra that admits countable joins, i.e. for every countable family $(B_n)_{n\in\mathbb{N}}$ the join $\bigvee_{n\in\mathbb{N}} B_n \in B$ is defined. In words of the partial order on B, every countable subset of B should have a supremum. An ultrafilter $\mathcal{G}$ in B is called *countably incomplete* if there exists a countable family $(P_n)_{n\in\mathbb{N}}$ of elements of B such that $P_n \notin \mathcal{G}$ for every $n \in \mathbb{N}$, $\bigvee_{n\in\mathbb{N}} P_n = 1_B$ equals the top element of B and for all $m, n \in \mathbb{N}$ we have that $m \neq n$ implies $P_m \wedge P_n = 0_B$.
This translates in the following way to our main examples of Boolean algebras: An ultrafilter $\mathcal{F}$ on a set Λ is countably incomplete if and only if there exists a countable partition $(P_n)_{n\in\mathbb{N}}$ of Λ such that $P_n \notin \mathcal{F}$ for every $n \in \mathbb{N}$. It is shown in [6, Theorem 6.1.4] that for every infinite set Λ there exists a countably incomplete ultrafilter on Λ.
An ultrafilter $\mathcal{U}$ in the Boolean algebra $\mathcal{B} = \prod_{\lambda\in\Lambda} \mathcal{P}(\max(D_\lambda))$ is countably incomplete if and only if there exists a family $(P_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}$ such that $P_n \notin \mathcal{U}$ for every $n \in \mathbb{N}$,

$\bigvee_{n\in\mathbb{N}} P_n = 1_{\mathcal{B}} = (\max(D_\lambda))_{\lambda\in\Lambda}$ and for all $m, n \in \mathbb{N}$ we have that $m \neq n$ implies $P_m \wedge P_n = 0_{\mathcal{B}}$. The family $(P_n)_{n\in\mathbb{N}}$ is called a *partition* of $1_{\mathcal{B}}$.

Lemma 3.6. If $\mathcal{F}$ is a countably incomplete ultrafilter on Λ, then every ultrafilter $\mathcal{U}$ in $\mathcal{B}$ with $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$ is countably incomplete.

Proof. Let $(P_n)_{n\in\mathbb{N}}$ be a partition of Λ such that $P_n \notin \mathcal{F}$ for all $n \in \mathbb{N}$. Define $Q^{(n)} := (Q^{(n)}_\lambda)_{\lambda\in\Lambda} \in \mathcal{B}$ for each $n \in \mathbb{N}$, where $Q^{(n)}_\lambda = \max(D_\lambda)$ if $\lambda \in P_n$ and $Q^{(n)}_\lambda = \emptyset$ else. Clearly, $\bigvee_{n\in\mathbb{N}} Q^{(n)} = 1_{\mathcal{B}}$ and $Q^{(m)} \wedge Q^{(n)} = 0_{\mathcal{B}}$ for all $m, n \in \mathbb{N}$ with $m \neq n$.
Assume that $Q^{(n)}_\lambda \in \mathcal{U}$ for some $n \in \mathbb{N}$. Then $P_n = \{\lambda \in \Lambda \mid Q^{(n)}_\lambda \neq \emptyset\} \in \mathcal{F}_{\mathcal{U}} = \mathcal{F}$, which is a contradiction. □

From now on, we again fix an ultrafilter $\mathcal{U}$ in $\mathcal{B}$ and set $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$ the induced ultrafilter on Λ.

Lemma 3.7. Let $\mathcal{F}$ be countably incomplete and $g, h \in S$ such that for all $\lambda \in \Lambda$ and for all $P \in \max(D_\lambda)$ we have $g_{\lambda,P} > 0$ and $h_{\lambda,P} > 0$.

(1) If $g \ll h$, then there exists some $k \in S$ such that $g \ll k \ll h$.
(2) If $g \ll \infty = (\infty)_{\lambda\in\Lambda}$, then there exists some $k \in S$ such that $g \ll k \ll \infty$.

Proof. (2) follows immediately by setting $h = \infty$ in (1).
To show (1), we can assume without loss of generality that $g_{\lambda,P} < h_{\lambda,P}$ for all $\lambda \in \Lambda$ and $P \in \max(D_\lambda)$. We define the following two complementary elements of the Boolean algebra $\mathcal{B}$:

$$V = (V_\lambda), \text{ where } V_\lambda = \{P \in \max(D_\lambda) \mid \forall n \in \mathbb{N}\ n \cdot g_{\lambda,P} < h_{\lambda,P}\} \text{ and}$$
$$W = (W_\lambda), \text{ where } W_\lambda = \{P \in \max(D_\lambda) \mid \exists N \in \mathbb{N}\ N \cdot g_{\lambda,P} \geq h_{\lambda,P}\}.$$

Since $\mathcal{U}$ is an ultrafilter in $\mathcal{B}$, we either have $V \in \mathcal{U}$ or $W \in \mathcal{U}$. Assume that $V \in \mathcal{U}$. Since $\mathcal{U}$ is countably incomplete by Lemma 3.6, we can choose a partition $(P_n)_{n\in\mathbb{N}}$ of $1_{\mathcal{B}}$ such that $P_n \notin \mathcal{U}$ for all $n \in \mathbb{N}$. By setting $V^{(n)} = P_n \wedge V$ for each $n \in \mathbb{N}$, we get a partition $(V^{(n)})_{n\in\mathbb{N}}$ of V such that $V^{(n)} \notin \mathcal{U}$ for every $n \in \mathbb{N}$ (in the sense that it is a partition of the top element V in the subalgebra $\mathcal{B}_V := \{Y \in \mathcal{B} \mid Y \leq V\}$). For $\lambda \in \Lambda$ and $P \in \max(D_\lambda)$, we define $k_{\lambda,P} = n \cdot g_{\lambda,P}$ if $P \in V^{(n)}_\lambda$ and $k_{\lambda,P} = g_{\lambda,P}$ if $P \notin V_\lambda$. Then clearly $k \ll h$.
To see that $g \ll k$, let $U \in \mathcal{U}$ and $n \in \mathbb{N}$. Then $U \wedge V \in \mathcal{U}$ and therefore there exists some $N > n$ such that $U \wedge V^{(N)} \neq 0_{\mathcal{B}}$. (Indeed, if for all $N > n$ we would have that $U \wedge V^{(N)} = \emptyset$, then $(U \wedge V^{(1)}) \vee \ldots \vee (U \wedge V^{(n)}) = U \wedge V \in \mathcal{U}$. Therefore there would exist some $i \in \{1, \ldots, n\}$ such that $U \wedge V^{(i)} \in \mathcal{U}$ and therefore $V^{(i)} \in \mathcal{U}$, which is a contradiction.) Pick some $\lambda \in \Lambda$ and $P \in U_\lambda \cap V^{(N)}_\lambda$. Then $k_{\lambda,P} = N \cdot g_{\lambda,P} > n \cdot g_{\lambda,P}$, so $g \ll k$.
Now consider the case where $W \in \mathcal{U}$. For each $\lambda \in \Lambda$ and $P \in W_\lambda$, there exists some $N > 1$ such that $h_{\lambda,P} \leq N \cdot g_{\lambda,P}$, so we can pick $N_{\lambda,P} \geq 1$ such that $N_{\lambda,P} \cdot g_{\lambda,P} < h_{\lambda,P} \leq (N_{\lambda,P} + 1) \cdot g_{\lambda,P}$. Define $k_{\lambda,P} = [N_{\lambda,P}/\log(N_{\lambda,P})] \cdot g_{\lambda,P}$, where $[x]$ denotes the floor function evaluated at $x \in \mathbb{R}$ and $[N_{\lambda,P}/0] := \infty$. For $P \notin W_\lambda$ let $k_{\lambda,P} = g_{\lambda,P}$.
Let $Y \in \mathcal{U}$ and $n \in \mathbb{N}$. We have to show the following two assertions:

(i) There exist $\lambda \in \Lambda$ and $P \in Y_\lambda$ such that $n \cdot g_{\lambda,P} < k_{\lambda,P}$.
(ii) There exist $\lambda \in \Lambda$ and $P \in Y_\lambda$ such that $n \cdot k_{\lambda,P} < h_{\lambda,P}$.

We can assume without loss of generality that $Y \leq W$. First of all, note that $\{N_{\lambda,P} \mid \lambda \in \Lambda, P \in Y_\lambda\}$ is unbounded, because $g \ll h$. It follows that the sets

$$S_{(i)} := \{[N_{\lambda,P}/\log(N_{\lambda,P})] \mid \lambda \in \Lambda, P \in Y_\lambda\} \text{ and}$$
$$S_{(ii)} := \{\log(N_{\lambda,P}) \mid \lambda \in \Lambda, P \in Y_\lambda\}$$

are also unbounded. To show (i), we use that $S_{(i)}$ is unbounded and pick $\lambda \in \Lambda$, $P \in Y_\lambda$ such that $n < [N_{\lambda,P}/\log(N_{\lambda,P})]$. It follows that $n \cdot g_{\lambda,P} < [N_{\lambda,P}/\log(N_{\lambda,P})] \cdot g_{\lambda,P} = k_{\lambda,P}$.
For the proof of (ii), we can pick $\lambda \in \Lambda$, $P \in Y_\lambda$ such that $n < \log(N_{\lambda,P})$. It follows that $n \cdot [N_{\lambda,P}/\log(N_{\lambda,P})] \leq n \cdot N_{\lambda,P}/\log(N_{\lambda,P}) < n \cdot N_{\lambda,P}/n = N_{\lambda,P}$. Hence $n \cdot k_{\lambda,P} = n \cdot [N_{\lambda,P}/\log(N_{\lambda,P})] \cdot g_{\lambda,P} < N_{\lambda,P} \cdot g_{\lambda,P} < h_{\lambda,P}$. □

Theorem 3.8. *Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of Prüfer domains and set $R = \prod_{\lambda\in\Lambda} D_\lambda$. Let $\mathcal{F}$ be a countably incomplete ultrafilter on Λ and $\mathcal{U}$ be an ultrafilter in the Boolean algebra $\mathcal{B} = \prod_{\lambda\in\Lambda} \mathcal{P}(\max(D_\lambda))$ such that $\mathcal{F}$ equals the unique induced ultrafilter $\mathcal{F}_\mathcal{U}$ on Λ and such that there exist $Y \in \mathcal{U}$ and $N \in \mathbb{N}$ with $|Y_\lambda| \leq N$ for all $\lambda \in \Lambda$ (e.g. let all D_λ be semilocal with a uniform bound on the cardinalities of $\max(D_\lambda)$).*
Then for every prime ideal $P \subseteq R$ with $(0)_\mathcal{F} \subsetneqq P$ there exists some prime ideal $Q \subseteq R$ such that $(0)_\mathcal{F} \subsetneqq Q \subsetneqq P$.
In particular, every prime ideal of R strictly containing $(0)_\mathcal{F}$ is of infinite height.

Proof. Let $I \subseteq S$ such that $P = \bigcup_{g\in I}(\mathcal{U})^g$, which exists by Theorem 3.4. Since $(0)_\mathcal{F} \subseteq (\mathcal{U})^g$ for all $g \in I$, there must exist some $g \in I$ such that $(\mathcal{U})^\infty = (0)_\mathcal{F} \subsetneqq (\mathcal{U})^g$. It follows by Lemma 3.5(1) that $g \ll \infty$. So by Lemma 3.7(2) we have that there exists some $h \in S$ with $g \ll h \ll \infty$. Lemma 3.5(2) implies that $(0)_\mathcal{F} \subsetneqq (\mathcal{U})^h \subsetneqq (\mathcal{U})^g \subseteq P$. The assertion follows by setting $Q = (\mathcal{U})^h$. □

We close with an example of a Prüfer domain in which every non-zero prime ideal is of infinite height: Let Λ be an infinite set and $(D_\lambda)_{\lambda\in\Lambda}$ be a family of semilocal Prüfer domains with the property that there exists $N \in \mathbb{N}$ such that $|\max(D_\lambda)| \leq N$ for all $\lambda \in \Lambda$. Moreover, let $\mathcal{F}$ be a countably incomplete ultrafilter on Λ (which exists by [6, Theorem 1.6.4]). Then by Theorem 3.8, every non-minimal prime ideal of $R = \prod_{\lambda\in\Lambda} D_\lambda$ containing $(0)_\mathcal{F}$ is of infinite height. As noted before, the ring theoretical property of being a Prüfer domain is preserved by ultraproducts.
So the ultraproduct $R^* = \prod^\mathcal{F}_{\lambda\in\Lambda} D_\lambda \cong R/(0)_\mathcal{F}$ is a Prüfer domain in which every non-zero prime ideal is of infinite height. If every D_λ is local, then R^* is in addition a valuation domain. If every D_λ is non-local, then so is R^*.

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