SETS OF LENGTHS OF FACTORIZATIONS OF INTEGER-VALUED POLYNOMIALS ON DEDEKIND DOMAINS WITH FINITE RESIDUE FIELDS

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Abstract. Let D be a Dedekind domain with infinitely many maximal ideals, all of finite index, and K its quotient field. Let $\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}$ be the ring of integer-valued polynomials on D.

Given any finite multiset $\{k_1, \ldots, k_n\}$ of integers greater than 1, we construct a polynomial in $\mathrm{Int}(D)$ which has exactly n essentially different factorizations into irreducibles in $\mathrm{Int}(D)$, the lengths of these factorizations being $k_1, \ldots, k_n$. We also show that there is no transfer homomorphism from the multiplicative monoid of $\mathrm{Int}(D)$ to a block monoid.

1. Introduction

By factorization we mean an expression of an element of a ring as a product of irreducible elements. Until not so long ago, the fact that such a factorization, if it exists, need not be unique, was seen as a pathology. When mathematicians were shocked to find that uniqueness of factorization does not hold in rings of integers in number fields, they did not immediately study the details of this non-uniqueness, but moved on to unique factorization of ideals into prime ideals. Non-uniqueness of factorization was avoided, whenever possible.

Only in the last few decades, some mathematicians, notably Geroldinger and Halter-Koch [9], came around to the view that the precise details of non-uniqueness of factorization actually are a fascinating topic: the underlying phenomena give a lot of information about the arithmetic of a ring.

One important object of study is the set of lengths of factorizations of a fixed element, cf. [8]. The length of a factorization is the number of irreducible factors, and the set of lengths of an element is the set of all natural numbers that occur as lengths of factorizations of the element. Geroldinger and Halter-Koch [9] found that the sets of lengths of algebraic integers exhibit a certain structure.

In stark contrast to this, we show in Section 4 that every finite set of natural numbers not containing 1 occurs as the set of lengths of a polynomial in the ring of integer-valued polynomials on D,

$$\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\},$$

where D is a Dedekind domain with infinitely many maximal ideals, all of them of finite index, and K denotes the quotient field of D. The special case of $D = \mathbb{Z}$ has been shown by Frisch [6].

The study of non-uniqueness of factorization has mostly concentrated on Krull monoids so far. Krull monoids are characterized by having a “divisor theory”. The multiplicative monoid $D \setminus \{0\}$ of an integral domain D is Krull exactly if D is a Krull ring, cf. [9].

The rings $\mathrm{Int}(D)$ for which we study non-uniqueness of factorization are not Krull, but Prüfer, cf. [4, 14]. All factorizations of a single polynomial in $\mathrm{Int}(D)$, however, take place in a Krull monoid, namely, in the divisor-closed submonoid of $\mathrm{Int}(D)$ generated by f.

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Following Reinhart [16], we call this monoid, consisting of all divisors in $\mathrm{Int}(D)$ of all powers of f, the *monadic submonoid generated by* f. That all monadic submonoids of $\mathrm{Int}(D)$ are Krull was shown by Reinhart [16] in the case where D is a unique factorization domain, and, by a different method, by Frisch [7] in the case where D is a Krull ring. Thus, our Theorem 1, concerning non-unique factorization in the Prüfer ring $\mathrm{Int}(D)$, also serves to show that quite wild factorization behavior is possible in Krull monoids.

Among Krull monoids, the best studied ones are multiplicative monoids of rings of algebraic integers. We should keep in mind, however, that the multiplicative monoids of rings of algebraic integers are very special, in that unique factorization of ideals always lurks in the background. In technical terms this means that there is a transfer homomorphism to a block monoid.

In Section 5, we show that there is no transfer homomorphism to a block monoid from the multiplicative monoid of $\mathrm{Int}(D)$. This is relevant for two reasons: Firstly, because the rings of whose multiplicative monoid it is known that it does not admit such a transfer homomorphism are few and far between, see [5, 10, 11]; and secondly, because most, if not all, results so far concerning arbitrary finite sets occurring as sets of lengths have been obtained using transfer homomorphisms to block monoids [13].

Our main results are in Sections 4 and 5; in Section 2 we introduce the necessary notation and Section 3 contains some useful lemmas.

2. Preliminaries

We start with a short review of some elementary facts on factorizations, Dedekind domains and integer-valued polynomials, and introduce some notation.

Factorizations. We define here only the notions that we need throughout this paper, and refer to the monograph by Geroldinger and Halter-Koch [9] for a systematic introduction to non-unique factorizations.

Let R be a commutative ring with identity and r, $s \in R$.

(i) If r is a non-zero non-unit, we say r is *irreducible* in R if it cannot be written as the product of two non-units of R.

(ii) A *factorization* of r in R is an expression

$$r = a_1 \cdots a_n \tag{1}$$

where $n \geq 1$ and a_i is irreducible in R for $1 \leq i \leq n$.

(iii) The number n of irreducible factors is called the *length* of the factorization in (1).

(iv) The *set of lengths* of r is the set of all natural numbers n such that r has a factorization of length n.

(v) We say r and s are *associated* in R if there exists a unit $u \in R$ such that $r = us$. We denote this by $r \sim s$.

(vi) Two factorizations of the same element,

$$r = a_1 \cdots a_n = b_1 \cdots b_m, \tag{2}$$

are called *essentially the same* if $n = m$ and, after reindexing, $a_j \sim b_j$ for $1 \leq j \leq m$. If this is not the case, the factorizations in (2) are called *essentially different.*

Dedekind domains. Recall that an integral domain D is a *Dedekind domain* if and only if every non-zero ideal is a product of prime ideals. This is equivalent to every non-zero ideal being invertible. It is also equivalent to D being a Noetherian domain such that the localization at every non-zero maximal ideal is a discrete valuation domain. And it is further equivalent to the following list of properties:

(i) D is Noetherian

(ii) D is integrally closed

(iii) $\dim(D) \leq 1$

From now on, we only consider Dedekind domains that are not fields. For a Dedekind domain D with quotient field K, let max-spec(D) denote the set of maximal ideals of D. Every prime ideal $P \in$ max-spec(D) defines a discrete valuation v_P by $\mathsf{v}_P(a) = \max\{n \in \mathbb{Z} \mid a \in P^n\}$ for $a \in K \setminus \{0\}$. v_P is called the *P-adic valuation* on K.

For a non-zero ideal I of D, let $\mathsf{v}_P(I) = \min\{\mathsf{v}_P(a) \mid a \in I\}$. This is compatible with the definition of $\mathsf{v}_P(a)$ for $a \in K \setminus \{0\}$, in the sense that $\mathsf{v}_P(aD) = \mathsf{v}_P(a)$. With this notation, the factorization of I into prime ideals is

$$I = \prod_{P \in \text{max-spec}(D)} P^{\mathsf{v}_P(I)} \tag{3}$$

Note that $\mathsf{v}_P(I) > 0$ is equivalent to $I \subseteq P$. There are only finitely many prime overideals of I in D and hence the product in Equation (3) is finite.

For two ideals I and J of D, $I \subseteq J$ is equivalent to $\mathsf{v}_P(J) \leq \mathsf{v}_P(I)$ for all $P \in$ max-spec(D). Note that $I \subseteq J$ is equivalent to the fact that there exists an ideal L of D such that $JL = I$, in which case we say that J divides I and write $J \mid I$. This last equivalence is often summarized as "to contain is to divide."

For a thorough introduction to Dedekind domains, we refer to Bourbaki [1, Ch. VII, § 2].

Dedekind domains with finite residue fields. Let D be a Dedekind domain. For a maximal ideal P with finite residue field we write $\|P\|$ for $|D/P|$ and call this number the *index of* P. In what follows we will only consider Dedekind rings with infinitely many maximal ideals, all of whose residue fields are finite. We will frequently use the fact that there are only finitely many maximal ideals of each individual finite index. This holds in every Noetherian domain, as Samuel [17] has shown; see also Gilmer [12].

We include a short proof by F. Halter-Koch for the special case of Dedekind domains.

Proposition 2.1 (Samuel [17], Gilmer [12]). *Let D be a Dedekind domain. Then for each given $q \in \mathbb{N}$, there are at most finitely many maximal ideals P of D with $\|P\| = q$.*

Proof (Halter-Koch, personal communication). Suppose that for some $q \geq 2$ there exist infinitely many prime ideals of index q, and let $0 \neq a \in D$. Then there exist infinitely many prime ideals P of D such that $\|P\| = q$ and $a \notin P$. For each such prime ideal P we obtain $a^{q-1} \equiv 1 \mod P$, hence $a^{q-1} - 1 \in P$ and thus $a^{q-1} = 1$. So, every non-zero element of D is a $(q-1)$-st root of unity. Impossible! □

Integer-valued polynomials. If D is a domain with quotient field K, the *ring of integer-valued polynomials* on D is defined as

$$\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}.$$

Every non-zero $f \in K[x]$ can be written as a quotient $f = \frac{g}{b}$ where $g \in D[x]$ and $b \in D \setminus \{0\}$. Clearly, $f = \frac{g}{b}$ is an element of $\mathrm{Int}(D)$ if and only if $b \mid g(a)$ for all $a \in D$.

Definition 2.2. Let D be a domain and $g \in \mathrm{Int}(D)$. The *fixed divisor* of g is the ideal $\mathsf{d}(g)$ of D generated by the elements $g(a)$ with $a \in D$:

$$\mathsf{d}(g) = (g(a) \mid a \in D)$$

We say that g is *image primitive* if $\mathsf{d}(g) = D$. By abuse of notation, this is also denoted $\mathsf{d}(g) = 1$.

Remark 2.3. Let D be a domain and K its quotient field.

(i) If $g \in D[x]$ and $b \in D \setminus \{0\}$, then $\frac{g}{b}$ is an element of $\mathrm{Int}(D)$ if and only if $\mathsf{d}(g) \subseteq bD$.
(ii) If $g \in D[x]$ and P a prime ideal of D such that $\mathsf{d}(g) \subseteq P$ then $g \in P[x]$ or $[D\colon P] \leq \deg(g)$.
(iii) If $f, g \in \mathrm{Int}(D)$, then $\mathsf{d}(fg) \subseteq \mathsf{d}(f)\mathsf{d}(g)$.
(iv) If $g \in D[x]$ is irreducible in $K[x]$, then every factorization of g in $\mathrm{Int}(D)$ as a product of two (not necessarily irreducible) elements is of the form $c\frac{g}{c}$ with $c \in D$ and $\mathsf{d}(g) \subseteq cD$.
(v) If $g \in D[x]$ is irreducible in $K[x]$ and $\mathsf{d}(g) = D$, then g is irreducible in $\mathrm{Int}(D)$.

For a general introduction to integer-valued polynomials we refer to the monograph by Cahen and Chabert [2] and to their more recent survey paper [3].

3. Auxiliary results

In this section we develop tools to construct, first, split polynomials in $D[x]$ with prescribed fixed divisor (Lemma 3.2), then, irreducible polynomials in $D[x]$ with prescribed fixed divisor (Lemma 3.3), and, finally, polynomials of a special form whose essentially different factorizations in $\mathrm{Int}(D)$ we have complete control over (Lemma 3.7).

Remark 3.1. In the following, we want to consider the multiplicity of roots of polynomials. For this purpose, we introduce some notation for multisets. Let $m_S(a)$ denote the multiplicity of an element a in a multiset S (with $m_S(a) = 0$ if $a \notin S$). For multisets S and T, let $S \uplus T$ denote the collection of elements a in the union of the sets underlying S and T with multiplicities $m_{S\uplus T}(a) = m_S(a) + m_T(a)$ (the disjoint union of S and T). Note that $|S \uplus T| = |S| + |T|$.

Lemma 3.2. *Let D be a domain, $\mathcal{T} \subseteq D$ a finite multiset and $f = \prod_{r\in\mathcal{T}}(x - r)$. If Q is a non-zero prime ideal of D, then $\mathsf{d}(f) \subseteq Q$ if and only if $\mathcal{T}$ contains a complete system of residues modulo Q.*

Furthermore, if D is a Dedekind domain and $\mathcal{T} = \mathcal{T}_0 \uplus \biguplus_{i=1}^{e} \mathcal{T}_i$ such that:

(i) For all $1 \leq i \leq e$, $\mathcal{T}_i$ is a complete system of residues modulo Q and the respective representatives of the same residue class in each $\mathcal{T}_i$ are congruent modulo Q^2,

(ii) There exists $z \in D$ such that for all $s \in \mathcal{T}_0$, $s \not\equiv z \mod Q$,

then $\mathsf{v}_Q(\mathsf{d}(f)) = e$.

Proof. If $\mathcal{T}$ does not contain a complete system of residues modulo Q, then there exists an element $a \in D$ such that $a \not\equiv r \mod Q$ for all $r \in \mathcal{T}$. This implies $f(a) = \prod_{r\in\mathcal{T}}(a - r) \notin Q$, hence $\mathsf{d}(f) \nsubseteq Q$.

Conversely, if $\mathcal{T}$ contains a complete system of residues modulo Q then, for all $a \in D$, there exists $r \in \mathcal{T}$ such that $a \equiv r \mod Q$. This implies $f(a) = \prod_{r\in\mathcal{T}}(a - r) \in Q$ for all $a \in D$ and hence $\mathsf{d}(f) \subseteq Q$.

Now assume that D is a Dedekind domain and $\mathcal{T} = \biguplus_{i=1}^{e} \mathcal{T}_i \uplus S$ such that (i) and (ii) hold. If $f_i = \prod_{r\in\mathcal{T}_i}(x-r)$ for $1 \leq i \leq e$ and $g = \prod_{s\in\mathcal{T}_0}(x-s)$, then $f = (\prod_{i=1}^{e} f_i)g$. Since $\mathcal{T}_i$ is a complete system of residues modulo Q, it follows that $\mathsf{v}_Q(f_i(a)) \geq 1$ for all $a \in D$. Therefore, for all $a \in D$,

$$\mathsf{v}_Q(f(a)) = \sum_{i=1}^{e} \mathsf{v}_Q(f_i(a)) + \mathsf{v}_Q(g(a)) \geq e \tag{4}$$

For $1 \leq i \leq e$, let $a_i \in \mathcal{T}_i$ with $a_i \equiv z \mod Q$. Since the elements a_i are in the same residue class modulo Q^2, there exists $d \in D$ in the same residue class modulo Q as z and all the a_i, but in a different residue class modulo Q^2 from all the a_i.

For such a d, then $\mathsf{v}_Q(f_i(d)) = 1$ for all $1 \leq i \leq e$ and $\mathsf{v}_Q(g(d)) = 0$, since for all $s \in \mathcal{T}_0$, $s \not\equiv z \equiv d \mod Q$. Therefore

$$\mathsf{v}_Q(f(d)) = \sum_{i=1}^{e} \mathsf{v}_Q(f_i(d)) + \mathsf{v}_Q(g(d)) = e$$

which implies that $\mathsf{v}_Q(\mathsf{d}(f)) = e$. $\square$

Next, we need to discuss how to replace split monic polynomials in $D[x]$ by monic polynomials in $D[x]$ which are irreducible in $K[x]$, without changing the fixed divisors.

Lemma 3.3. *Let D be a Dedekind domain with infinitely many maximal ideals and K its quotient field. Let $I \neq \emptyset$ be a finite set and $f_i \in D[x]$ be monic polynomials for $i \in I$.*

Then, there exist monic polynomials $F_i \in D[x]$ for $i \in I$, such that

(i) $\deg(F_i) = \deg(f_i)$ for all $i \in I$,

(ii) the polynomials F_i are irreducible in $K[x]$ and pairwise non-associated in $K[x]$ and

(iii) for all subsets $J \subseteq I$ and all partitions $J = J_1 \uplus J_2$,

$$\mathsf{d}\left(\prod_{j\in J_1} f_j \prod_{j\in J_2} F_j\right) = \mathsf{d}\left(\prod_{j\in J} f_j\right).$$

Proof. Let $P_1, \ldots, P_n$ be all maximal ideals P of D with $\|P\| \leq \deg\left(\prod_{i\in I} f_i\right)$. Suppose the prime factorization of the fixed divisor of the product of the f_i is

$$\mathsf{d}\left(\prod_{i\in I} f_i\right) = \prod_{j=1}^{n} P_j^{e_j}.$$

Let $Q \in \text{max-spec}(D) \setminus \{P_1, \ldots, P_n\}$. Using the Chinese Remainder Theorem, we add elements to the coefficients of the f_i such that the resulting polynomials can be seen to be irreducible according to Eisenstein's irreducibility criterion with respect to Q, while retaining all relevant properties with respect to sufficiently high powers of the P_i.

Let f_{ik} denote the coefficient of x^k in f_i. For $i \in I$ and $0 \leq k < \deg(f_i)$, let $g_{ik} \in D$ such that

(i) $g_{ik} \in \prod_{j=1}^{n} P_j^{e_j+1}$ for all $0 \leq k < \deg(f_i)$.
(ii) $g_{ik} \equiv -f_{ik} \mod Q$ for all $0 \leq k < \deg(f_i)$ and
(iii) $g_{i0} \not\equiv -f_{i0} \mod Q^2$.

Since the g_{ik} satisfying the above conditions are only determined modulo $Q^2 \prod_{i=1}^{n} P_i^{e_i+1}$, there are infinitely many choices for each g_{ik}. We use this flexibility to implement that $g_{i0}+f_{i0} \neq g_{j0}+f_{j0}$ for $i \neq j$. Then, for $i \in I$, we set

$$F_i = f_i + \sum_{k=0}^{\deg(f_i)-1} g_{ik}x^k.$$

As the resulting F_i are monic and distinct, they are pairwise non-associated in $K[x]$.

According to Eisenstein's irreducibility criterion, the polynomials F_i are irreducible in $D[x]$ for $i \in I$, cf. [15, §29, Lemma 1]. Since the F_i are monic and D is integrally closed, it follows that the F_i are irreducible in $K[x]$ for all $i \in I$, cf. [1, Ch. 5, §1.3, Prop. 11].

By construction,

$$F_i \equiv f_i \mod \left(\prod_{j=1}^{n} P_j^{e_j+1}\right) D[x]$$

for all $i \in I$. Now, if $g(x)$ is the product of any selection of the polynomials f_i, and $G(x)$ the modified product in which some of the f_i have been replaced by F_i, then $g(x)$ is congruent to $G(x)$ modulo $\left(\prod_{j=1}^{n} P_j^{e_j+1}\right) D[x]$.

Hence, for all $a \in D$, $g(a) \equiv G(a)$ modulo $\left(\prod_{j=1}^{n} P_j^{e_j+1}\right)$ and, therefore,

$$\min_{a\in D} \mathsf{v}_P(G(a)) = \min_{a\in D} \mathsf{v}_P(g(a))$$

for all P that could conceivably divide the fixed divisor of $G(x)$ or $g(x)$ by Remark 2.3.(ii). This implies the last assertion of the Lemma, to the effect that substituting F_i for some or all of the f_i does not change the fixed divisor of a product. $\square$

Finally, the last two lemmas enable us to understand all essentially different factorizations of a certain type of polynomials in $\text{Int}(D)$.

Lemma 3.4. *Let D be a Dedekind domain with quotient field K and $f \in \text{Int}(D)$ of the following form:*

$$f = \frac{\prod_{i\in I} f_i}{c} \quad \text{with} \quad \mathsf{d}\left(\prod_{i\in I} f_i\right) = cD,$$

where c is a non-unit of D and for each $i \in I$, $f_i \in D[x]$ is irreducible in $K[x]$.

Let $\mathcal{P} \subseteq \text{max-spec}(D)$ be the finite set of prime ideal divisors of cD. If $f = g_1 \cdots g_m$ is a factorization of f into (not necessarily irreducible) non-units in $\text{Int}(D)$ *then each g_j is of the form*

$$g_j = a_j \prod_{i\in I_j} f_i,$$

where $\emptyset \neq I_j \subseteq I$ *and* $a_j \in K$*, such that* $I_1 \uplus \ldots \uplus I_m = I$*,* $a_1 \cdots a_m = c^{-1}$ *and*

(i) $\mathsf{v}_P(a_j) \leq 0$ *for all* $P \in \text{max-spec}(D)$ *and all* $1 \leq j \leq m$*; and*

(ii) $\mathsf{v}_P(a_j) = 0$ *for all* $P \in \text{max-spec}(D) \setminus \mathcal{P}$ *and all* $1 \leq j \leq m$*.*

Proof. Let $f = g_1 \cdots g_m$ be a factorization of f into (not necessarily irreducible) non-units in $\text{Int}(D)$. Since $\mathsf{d}(f) = 1$, no g_i is a constant, by Remark 2.3.(iv). Each factor g_j is, therefore, of the form

$$g_j = a_j \prod_{i \in I_j} f_i \tag{5}$$

where I_j is a non-empty subset of I and $a_j \in K$, such that $I_1 \uplus \ldots \uplus I_m = I$ and $a_1 \cdots a_m = c^{-1}$. Note that for all $P \in \text{max-spec}(D)$

$$\sum_{j=1}^{m} \mathsf{v}_P(a_j) = -\mathsf{v}_P(c). \tag{6}$$

Suppose $\mathsf{v}_P(a_t) > 0$ for some maximal ideal P and some $1 \leq t \leq m$. Then $\sum_{j \neq t} \mathsf{v}_P(a_j) < -\mathsf{v}_P(c)$.

Remark 2.3.(iii) and the fact that $\mathsf{v}_P\left(\mathsf{d}\left(\prod_{i \in I} f_i\right)\right) = \mathsf{v}_P(c)$ imply $\mathsf{v}_P\left(\mathsf{d}\left(\prod_{j \neq t} \prod_{i \in I_j} f_i\right)\right) \leq \mathsf{v}_P(c)$. But now

$$\mathsf{v}_P\left(\mathsf{d}\left(\prod_{j \neq t} g_j\right)\right) = \mathsf{v}_P\left(\mathsf{d}\left(\prod_{j \neq t} \prod_{i \in I_j} f_i\right)\right) + \sum_{j \neq t} \mathsf{v}_P(a_j) < 0,$$

which means that

$$\prod_{j \neq t} g_j \notin \text{Int}(D),$$

a contradiction. We have established that $\mathsf{v}_P(a_j) \leq 0$ for all $P \in \text{max-spec}(D)$ and all $1 \leq j \leq m$. Now Equation (6) and the fact that $\mathsf{v}_P(c) = 0$ for all $P \notin \mathcal{P}$ imply $\mathsf{v}_P(a_j) = 0$ for all $P \notin \mathcal{P}$ and all $1 \leq j \leq m$. $\square$

Definition 3.5. Let D be a Dedekind domain, $P \in \text{max-spec}(D)$ and $f_i \in D[x]$ for a finite set $I \neq \emptyset$. We say f_k is *indispensable for* P (among the polynomials f_i with $i \in I$) if there exists an element $z \in D$ such that $\mathsf{v}_P(f_k(z)) > 0$ and $\mathsf{v}_P(f_i(z)) = 0$ for all $i \neq k$.

Remark 3.6. Note that f_k indispensable for P (among the polynomials f_i with $i \in I$) implies: for every $J \subseteq I$

$$\mathsf{v}_P\left(\mathsf{d}\left(\prod_{i \in J} f_i\right)\right) > 0 \Longrightarrow k \in J.$$

Lemma 3.7. *Let* D *be a Dedekind domain with quotient field* K *and* $f \in \text{Int}(D)$ *of the following form:*

$$f = \frac{\prod_{i \in I} f_i}{c} \quad \text{with} \quad \mathsf{d}\left(\prod_{i \in I} f_i\right) = cD,$$

where c *is a non-unit of* D *and for each* $i \in I$*,* $f_i \in D[x]$ *is irreducible in* $K[x]$*. Let* $\mathcal{P} \subseteq \text{max-spec}(D)$ *be the finite set of prime ideal divisors of* cD*.*

Suppose, for each $P \in \mathcal{P}$*,* Λ_P *is a subset of* I *such that* f_i *is indispensable for* P *for each* $i \in \Lambda_P$*. Let* $\Lambda = \bigcup_{P \in \mathcal{P}} \Lambda_P$*.*

If $\bigcap_{P \in \mathcal{P}} \Lambda_P \neq \emptyset$ *then all essentially different factorizations of* f *into irreducibles in* $\text{Int}(D)$ *are given by:*

$$\frac{\left(\prod_{i \in \Lambda \cup J_1} f_i\right)}{c} \cdot \prod_{j \in J_2} f_j$$

(each f_j *with* $j \in J_2$ *counted as an individual factor), where* $I = \Lambda \uplus J_1 \uplus J_2$ *such that* J_1 *is minimal with* $\mathsf{d}\left(\prod_{i \in \Lambda \cup J_1} f_i\right) = cD$*.*

Proof. Let $f = g_1 \cdots g_m$ be a factorization of f into (not necessarily irreducible) non-units in $\mathrm{Int}(D)$. As in Lemma 3.4,

$$g_j = a_j \prod_{i \in I_j} f_i \tag{7}$$

where I_j is a non-empty subset of I and $a_j \in K$, such that $I_1 \uplus \ldots \uplus I_m = I$ and $a_1 \cdots a_m = c^{-1}$. Furthermore, $\mathsf{v}_P(a_j) \leq 0$ for all $P \in \text{max-spec}(D)$ and all $1 \leq j \leq m$ and $\mathsf{v}_P(a_j) = 0$ for all $P \notin \mathcal{P}$ and all $1 \leq j \leq m$.

We know there exists a polynomial f_{i_0} that is indispensable for all $P \in \mathcal{P}$. We may assume that $i_0 \in I_1$. By the definition of indispensable polynomial, $\mathsf{v}_P\left(\mathsf{d}\left(\prod_{i\in I_j} f_i\right)\right) = 0$, for $2 \leq j \leq m$ and all $P \in \mathcal{P}$. From this and the fact that $g_j = a_j \prod_{i \in I_j} f_i$ is in $\mathrm{Int}(D)$, we infer that $\mathsf{v}_P(a_j) = 0$ for all $2 \leq j \leq m$ and all $P \in \mathcal{P}$. We have shown that $a_2, \ldots, a_m$ are units of D.

Now $u = a_2 \cdots a_m$ is a unit of D such that $a_1 u = c^{-1}$. Since $g_1 \in \mathrm{Int}(D)$, we must have

$$\mathsf{v}_P\left(\mathsf{d}\left(\prod_{i\in I_1} f_i\right)\right) = \mathsf{v}_P(c)$$

for all $P \in \mathcal{P}$ and, by Remark 3.6, $\Lambda \subseteq I_1$.

So far we have shown that every factorization $f = g_1 \cdots g_m$ of f into (not necessarily irreducible) non-units of $\mathrm{Int}(D)$ is – up to reordering of factors and multiplication of factors by units in D – the same as one of the following:

$$\frac{\left(\prod_{i\in\Lambda\cup J_1} f_i\right)}{c} \cdot \left(\prod_{j\in I_2} f_j\right) \cdots \left(\prod_{j\in I_m} f_j\right), \tag{8}$$

where $I = I_1 \uplus \cdots \uplus I_m$ and $I_1 = \Lambda \uplus J_1$.

It remains to characterize, among the factorizations of the above form, those in which all factors are irreducible in $\mathrm{Int}(D)$.

Since $\mathsf{d}(f) = D$, it is clear that $\mathsf{d}(g_j) = D$ for all $1 \leq j \leq m$, by Remark 2.3.(iii). By the same token, $\mathsf{d}(f_i) = D$ for all $i \in I_j$ with $j \geq 2$. Since the f_i are irreducible in $K[x]$, those of them with fixed divisor D are irreducible in $\mathrm{Int}(D)$, by Remark 2.3.(v). The criterion for each factor $g_j = \prod_{i\in I_j} f_i$ with $j \geq 2$ to be irreducible is, therefore, $|I_j| = 1$ for all $j \geq 2$.

Now, concerning the irreducibility of g_1, the same arguments that lead to Equation (8), applied to $g_1 = c^{-1}\left(\prod_{i\in\Lambda\cup J_1} f_i\right)$ instead of f, show that g_1 is irreducible in $\mathrm{Int}(D)$ if and only if we cannot split off any factors f_i with $i \in J_1$. This is equivalent to $\mathsf{d}\left(\prod_{i\in\Lambda\cup J} f_i\right) \neq cD$ for every proper subset $J \subsetneq J_1$, in other words, to J_1 being minimal such that $\mathsf{d}\left(\prod_{i\in\Lambda\cup J_1} f_i\right) = cD$. In this case we set $J_2 = \bigcup_{j=2}^m I_j$ and the assertion follows. □

Remark 3.8. When $|\mathcal{P}| > 1$, the hypothesis $\bigcap_{P\in\mathcal{P}} \Lambda_P \neq \emptyset$ in Lemma 3.7 can be replaced by a weaker condition:

Consider the prime ideals $P \in \mathcal{P}$ as vertices of an undirected graph G and let (P, Q) be an edge of G if and only if there exists a polynomial f_t which is indispensable for both P and Q. If G is a connected graph, then the conclusion of Lemma 3.7 holds. The proof of Lemma 3.7 generalizes readily.

4. Construction of polynomials with prescribed sets of lengths

We are now ready to prove the main result of this paper.

Theorem 1. *Let D be a Dedekind domain with infinitely many maximal ideals, all of them of finite index.*

Let $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n$ be natural numbers.

Then there exists a polynomial $H \in \mathrm{Int}(D)$ with exactly n essentially different factorizations into irreducible polynomials in $\mathrm{Int}(D)$, the length of these factorizations being $m_1 + 1, \ldots, m_n + 1$.

Proof. If $n = 1$, then $H(x) = x^{m_1+1} \in \mathrm{Int}(D)$ is a polynomial which has exactly one factorization, and this factorization has length $m_1 + 1$. From now on, assume $n \geq 2$.

First, we construct $H(x)$. Let $N = \left(\sum_{i=1}^{n} m_i\right)^2 - \sum_{i=1}^{n} m_i^2$ and P a prime ideal of D with $\|P\| > N+1$. Let $c \in D$ such that $\mathsf{v}_P(c) = 1$ and c is not contained in any maximal ideal of index 2.

Say the prime factorization of cD is $cD = PQ_1^{e_1} \cdots Q_t^{e_t}$. Let $\tau = (\|P\| - N)$ and σ the maximum of the following numbers: τ, and $e_i\|Q_i\|$ for $1 \leq i \leq t$.

We now choose two subsets of D: a set $\mathcal{R}$ of order N, and $\mathcal{S} = \{s_0, \ldots, s_{\sigma-1}\}$. Using the Chinese Remainder Theorem, we arrange that $\mathcal{R}$ and $\mathcal{S}$ have the following properties:

(i) $s_0 \equiv 0 \mod P$, and $\{s_0, \ldots, s_{\tau-1}\} \cup \mathcal{R}$ is a complete system of residues modulo P.
(ii) $s_i \equiv 0 \mod P$ for all $i \geq \tau$.
(iii) For each Q_i, $\mathcal{S}$ contains e_i disjoint complete systems of residues, in which the respective representatives of the same residue class in different systems are congruent modulo Q_i^2.
(iv) For each Q_i, no more than e_i elements of $\mathcal{S}$ are congruent to 1 modulo Q_i.
(v) For all $r \in \mathcal{R}$, $r \equiv 0 \mod \bigcap_{i=1}^{t} Q_i$.
(vi) $\mathcal{R} \cup \mathcal{S}$ does not contain a complete system of residues for any prime ideal Q of D other than P and $Q_1, \ldots, Q_t$.

We now assign indices to the elements of $\mathcal{R}$ as follows

$$\mathcal{R} = \{r_{(k,i,h,j)} \mid 1 \leq k, h \leq n, k \neq h, 1 \leq i \leq m_k, 1 \leq j \leq m_h\}.$$

This allows us to visualize the elements of $\mathcal{R}$ as entries in a square matrix B with $m = \sum_{i=1}^{n} m_i$ rows and columns, in which the positions in the blocks of a block-diagonal matrix with block sizes $m_1, \ldots, m_n$ are left empty, see Fig. 1.

The rows and columns of B are divided into n blocks each, such that the k-th block of rows consists of m_k rows, and similarly for columns. Now $r_{(k,i,h,j)}$ designates the entry in row (k, i), that is, in the i-th row of the k-th block of rows, and in column (h, j), that is, in the j-th column of the h-th block of columns. Since no element of $\mathcal{R}$ has row and column index in the same block, the positions of a block-diagonal matrix with blocks of sizes $m_1, \ldots, m_n$ are left empty.

For $1 \leq k \leq n$, let $I_k = \{(k, i) \mid 1 \leq i \leq m_k\}$ and set

$$I = \bigcup_{k=1}^{n} I_k.$$

Then

$$I = \{(k, i) \mid 1 \leq k \leq n, 1 \leq i \leq m_k\}$$

is the set of all possible row indices, or, equivalently, column indices.

For $(k, i) \in I_k$, let $B[k, i]$ be the set of all elements $r \in \mathcal{R}$ which are either in row or in column (k, i) of B, that is,

$$B[k, i] = \{r_{(k,i,h,j)} \mid (h, j) \in I \setminus I_k\} \cup \{r_{(h,j,k,i)} \mid (h, j) \in I \setminus I_k\} \tag{9}$$

In order to construct $H \in \mathrm{Int}(D)$, we set $s(x) = \prod_{i=0}^{\sigma-1}(x - s_i)$ and, for $(k, i) \in I$,

$$f_i^{(k)}(x) = \prod_{r \in B[k,i]} (x - r).$$

Then, let $S(x) \in D[x]$, and, for each $(k, i) \in I$, $F_i^{(k)}(x) \in D[x]$ be monic polynomials such as we know to exist by Lemma 3.3: irreducible in $K[x]$, pairwise non-associated in $K[x]$, with $\deg(S) = \deg(s)$ and $\deg(F_i^{(k)}) = \deg(f_i^{(k)})$, and such that, for every selection of polynomials from among s and $f_i^{(k)}$ for $(k, i) \in I$, the product of the polynomials has the same fixed divisor as the modified product in which s has been replaced by S and each $f_i^{(k)}$ by $F_i^{(k)}$. Now, let

$$G(x) = S(x) \prod_{(k,i) \in I} F_i^{(k)}(x) \quad \text{and} \quad H(x) = \frac{G(x)}{c}.$$

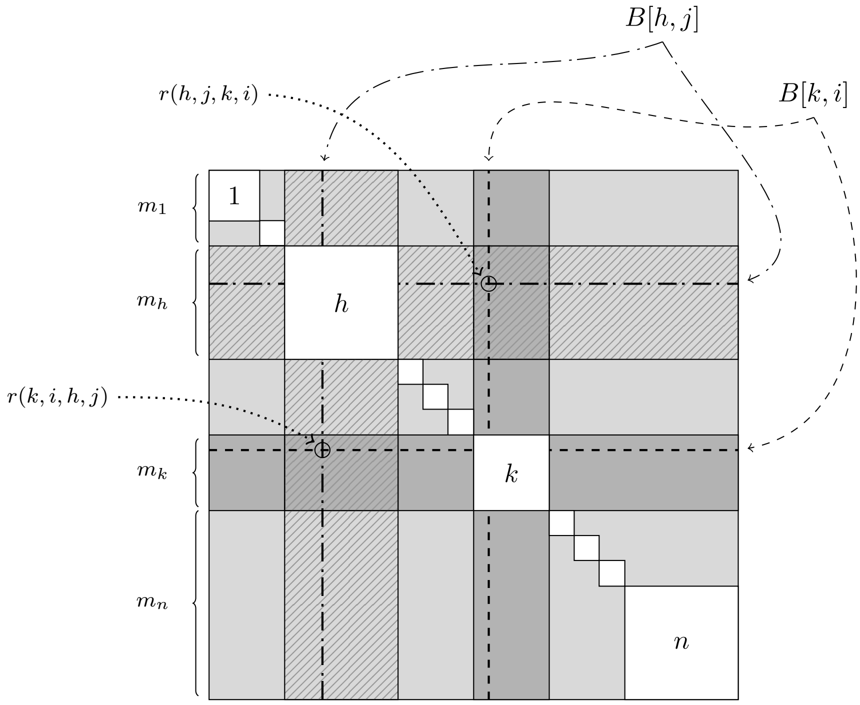

FIGURE 1. Say the k-th region of B consists of the positions with either column index or row index in the k-th block. Then the union of the entries in any $n-1$ different regions covers $\mathcal{R}$. A union of different $B[u,v]$, from which $B[k,i]$ and $B[h,j]$ for two different blocks $k \neq h$ are missing, however, does not cover $\mathcal{R}$, because $r_{(k,i,h,j)}$ and $r_{(h,j,k,i)}$ are not included.

Second, we show that $\mathsf{d}(G(x)) = cD$, which implies $H(x) \in \mathrm{Int}(D)$ and $\mathsf{d}(H(x)) = 1$. Note that

$$\mathsf{d}(G(x)) = \mathsf{d}\left(S(x)\prod_{(k,i)\in I} F_i^{(k)}(x)\right) = \mathsf{d}\left(s(x)\prod_{(k,i)\in I} f_i^{(k)}\right) = \mathsf{d}\left(\prod_{i=0}^{\sigma-1}(x-s_i)\prod_{r\in\mathcal{R}}(x-r)^2\right). \tag{10}$$

By construction, the multiset $\mathcal{R} \uplus \mathcal{R} \uplus \mathcal{S}$ contains a complete system of residues modulo P, and the residue class modulo P of $s_1 \in \mathcal{S}$ occurs only once among the elements of $\mathcal{R} \uplus \mathcal{R} \uplus \mathcal{S}$. Equation (10) and Lemma 3.2, applied to $Q = P$ and $\mathcal{T} = \mathcal{R} \uplus \mathcal{R} \uplus \mathcal{S}$, $e = 1$, and $z = s_1$, together imply that

$$\mathsf{v}_P\left(\mathsf{d}\left(S(x)\prod_{(k,i)\in I} F_i^{(k)}(x)\right)\right) = 1$$

One can argue similarly for Q_i, $1 \leq i \leq t$: The multiset $\mathcal{R} \uplus \mathcal{R} \uplus \mathcal{S}$ contains e_i disjoint complete systems of residues modulo Q_i in which the respective representatives of the same residue class in different systems are congruent modulo Q_i^2. No more than e_i elements of $\mathcal{R} \uplus \mathcal{R} \uplus \mathcal{S}$ are congruent 1 modulo Q_i, and these e_i elements are all in the same residue class modulo Q_i^2. By Lemma 3.2, applied to $Q = Q_i$, $\mathcal{T} = \mathcal{R} \uplus \mathcal{R} \uplus \mathcal{S}$, $e = e_i$ and $z = 1$, and Equation (10), it follows that

$$\mathsf{v}_{Q_i}\left(\mathsf{d}\left(S(x)\prod_{(k,i)\in I} F_i^{(k)}(x)\right)\right) = e_i \tag{11}$$

for $1 \le i \le t$. Since $\mathcal{R} \uplus \mathcal{R} \uplus \mathcal{S}$ does not contain a complete system of residues modulo any prime ideal other than P, $Q_1, \ldots, Q_t$, we conclude (by Lemma 3.2) that

$$\mathsf{d}(G(x)) = \mathsf{d}\left(S(x) \prod_{(k,i)\in I} F_i^{(k)}(x)\right) = PQ_1^{e_1} \cdots Q_t^{e_t} = cD.$$

This shows $H(x) \in \mathrm{Int}(D)$ and $\mathsf{d}(H(x)) = 1$.

Third, we prove that the essentially different factorizations of $H(x)$ into irreducibles in $\mathrm{Int}(D)$ are given by:

$$H(x) = F_1^{(h)}(x) \cdots F_{m_h}^{(h)}(x) \cdot \frac{S(x) \prod_{(k,i)\in I\setminus I_h} F_i^{(k)}(x)}{c} \tag{12}$$

where $1 \le h \le n$.

It follows from the properties of $\mathcal{R}$ and $\mathcal{S}$ that the polynomial $S(x)$ is indispensable for the prime ideals P and $Q_1, \ldots, Q_t$ (among the polynomials $S(x)$ and $F_i^{(k)}$ for $(k,i) \in I$).

Thus, by Lemma 3.7, the essentially different factorizations of $H(x)$ into irreducibles in $\mathrm{Int}(D)$ are given by:

$$H(x) = \frac{S(x) \prod_{(k,i)\in J} F_i^{(k)}(x)}{c} \prod_{(h,j)\in I\setminus J} F_j^{(h)}(x) \tag{13}$$

where $J \subseteq I$ is minimal such that $\mathsf{d}\left(S(x) \prod_{(k,i)\in J} F_i^{(k)}(x)\right) = cD$.

Since $\mathsf{v}_{Q_i}(\mathsf{d}(S(x))) = e_i$ by Lemma 3.2, the possible choices for $J \subseteq I$ only depend on the prime ideal P. For a subset $J \subseteq I$, let $\mathcal{B}_J = \biguplus_{(k,i)\in J} B[k,i]$. Then

$$\mathsf{d}\left(S(x) \prod_{(k,i)\in J} F_i^{(k)}(x)\right) = \mathsf{d}\left(\prod_{r\in\mathcal{S}}(x-r) \prod_{r\in\mathcal{B}_J}(x-r)\right) \tag{14}$$

and it follows from Lemma 3.2 that the fixed divisor in Equation (14) is equal cD if and only if $\mathcal{S} \uplus \mathcal{B}_J$ contains a complete set of residues modulo P which is in turn equivalent to $\mathcal{R} \subseteq \mathcal{B}_J$. This is the case if and only if there exists $1 \le h \le n$ with $I \setminus I_h \subseteq J$.

Therefore, $J \subseteq I$ is minimal with $\mathsf{d}\left(S(x) \prod_{(k,i)\in J} F_i^{(k)}(x)\right) = cD$ if and only if $J = I \setminus I_h$ for some $1 \le h \le n$. Hence, the essentially different factorizations of $H(x)$, given by Equation (13), are precisely the n essentially different factorizations stated in Equation (12), which are of lengths $m_1 + 1, \ldots, m_n + 1$. ☐

Corollary 4.1. *Let D be a Dedekind domain with infinitely many maximal ideals, all of them of finite index.*

Then every finite subset of $\mathbb{N} \setminus \{1\}$ is the set of lengths of a polynomial in $\mathrm{Int}(D)$.

Remark 4.2. Kainrath [13, Theorem 1] proved a similar result as Corollary 4.1 for Krull monoids H with infinite class group in which every divisor class contains a prime divisor. In his proof, he uses transfer mechanisms.

Corollary 5.1 in the following section will show that this technique is not applicable to the proof of either Theorem 1 or Corollary 4.1.

5. Not a transfer Krull domain

In this section we show that if D is a Dedekind domain with infinitely many maximal ideals, all of finite index, then there does not exist a transfer homomorphism from the multiplicative monoid $\mathrm{Int}(D) \setminus \{0\}$ to a block monoid. In the terminology introduced by Geroldinger [8], this means, $\mathrm{Int}(D)$ is not a *transfer Krull domain.*

We refer to [9, Definitions 2.5.5 & 3.2.1] for the definition of a block monoid and a transfer homomorphism, respectively. So far, there is only a small list of examples of naturally occurring rings R for which it has been shown that there is no transfer homomorphism from $R \setminus \{0\}$ to a block monoid, see [5, 10, 11].

In a block monoid, the lengths of factorizations of elements of the form $c\cdot d$ with c, d irreducible, c fixed, are bounded by a constant depending only on c, cf. [9, Lemma 6.4.4]. More generally, every monoid admitting a transfer homomorphism to a block monoid has this property; see [9, Proposition 3.2.3].

We now demonstrate for the irreducible element $c = x$ in $\mathrm{Int}(D)$ that the lengths of factorizations of elements of the form $c \cdot d$ with d irreducible in $\mathrm{Int}(D)$ are not bounded. We infer from this that there does not exist a transfer homomorphism from the multiplicative monoid $\mathrm{Int}(D) \setminus \{0\}$ to a block monoid.

Theorem 2. *Let D be a Dedekind domain with infinitely many maximal ideals, all of them of finite index.*

Then for every $n \geq 1$ there exist irreducible elements $H, G_1, \dots, G_{n+1}$ in $\mathrm{Int}(D)$ such that

$$xH(x) = G_1(x) \cdots G_{n+1}(x).$$

Proof. Let $P_1, \dots, P_n$ be distinct maximal ideals of D, none of them of index 2. By v_i we denote the discrete valuation associated to P_i. Let $c \in D$ such that $\mathsf{v}_i(c) = 1$ for $i = 1, \dots, n$, and c is not contained in any maximal ideal of D of index 2. Say the prime factorization of cD is $cD = P_1 \cdot \ldots \cdot P_n \cdot Q_1^{e_1} \cdot \ldots \cdot Q_m^{e_m}$, and define

$$N = \max\left(\{\|P_i\| \mid 1 \leq i \leq n\} \cup \{e_i\|Q_i\| \mid 1 \leq i \leq m\}\right).$$

Let $\mathcal{P} = \{P_i \mid 1 \leq i \leq n\}$, $\mathcal{P}_1 = \{Q_i \mid 1 \leq i \leq m\}$, and

$$\mathcal{P}_2 = \{Q \in \text{max-spec}(D) \setminus (\mathcal{P} \cup \mathcal{P}_1) \mid \|Q\| \leq N + n\}.$$

Let $\mathcal{R}$ be a subset of D of order N with the following properties (which can be realized by the Chinese Remainder Theorem):

(i) $\mathcal{R}$ contains an element $r_0 \in \left(\bigcap_{i=1}^n P_i\right) \cap \left(\bigcap_{i=1}^m Q_i^2\right)$.
(ii) No element of $\mathcal{R}$ other than r_0 is in any $P_i \in \mathcal{P}$.
(iii) For each $P_i \in \mathcal{P}$, $\mathcal{R}$ contains a complete system of residues modulo P_i.
(iv) For each $Q_i \in \mathcal{P}_1$, $\mathcal{R}$ contains e_i disjoint complete systems of residues, in which the respective representatives of the same residue class in different systems are congruent modulo Q_i^2;
(v) No more than e_i elements of $\mathcal{R}$ are in Q_i.
(vi) For all $Q \in \mathcal{P}_2$, all elements of $\mathcal{R}$ are contained in Q.

We set $\mathcal{B} = \mathcal{R} \setminus \{r_0\}$.

Also, let $a_1, \dots, a_n \in D$ with the following properties (which, again, can be realized by the Chinese Remainder Theorem):

(i) For all $i = 1, \dots, n$, $a_i \equiv 0 \mod P_i$.
(ii) For all $i = 1, \dots, n$, $a_i \equiv 1 \mod P_j$ for all $j \neq i$.
(iii) For all $Q \in \mathcal{P}_1$, $a_n \equiv 0 \mod Q^2$ and $a_i \equiv 1 \mod Q$ for all $1 \leq i < n$,
(iv) For all $Q \in \mathcal{P}_2$ and all $1 \leq i \leq n$, $a_i \equiv 0 \mod Q$.

Let $f(x) = \prod_{b \in \mathcal{B}}(x - b)$ and let $F(x) \in D[x]$ be monic and irreducible in $K[x]$ such that for every selection of polynomials from the set $\{x, f\} \cup \{(x - a_i) \mid 1 \leq i \leq n\}$ the product of the polynomials has the same fixed divisor as the modified product in which f has been replaced by F, as in Lemma 3.3.

Lemmas 3.3 and 3.2, applied to $\mathcal{T} = \mathcal{B} \cup \{a_1, \dots, a_n\}$ and each of the prime ideals in $\mathcal{P} \cup \mathcal{P}_1$, imply that

$$\mathsf{d}\left(F(x)\prod_{i=1}^n (x - a_i)\right) = \mathsf{d}\left(f(x)\prod_{i=1}^n (x - a_i)\right) = cD.$$

Similarly, Lemmas 3.3 and 3.2, applied to $\mathcal{T} = \mathcal{B} \cup \{0\}$ and each of the prime ideals in $\mathcal{P} \cup \mathcal{P}_1$, imply that

$$\mathsf{d}\left(xF(x)\right) = \mathsf{d}\left(xf(x)\right) = cD.$$

We set

$$H(x) = \frac{F(x)\prod_{j=1}^n (x - a_j)}{c} \quad \text{and} \quad G(x) = \frac{xF(x)}{c}.$$

Then $G(x)$ and $H(x)$ are elements of $\mathrm{Int}(D)$ with $\mathsf{d}(G(x)) = \mathsf{d}(H(x)) = 1$ such that

$$xH(x) = G(x)(x - a_1) \cdots (x - a_n).$$

It remains to show that $H(x)$ and $G(x)$ are irreducible in $\mathrm{Int}(D)$. Observe that x is indispensable for all $P \in \mathcal{P}$, and $F(x)$ is indispensable for all $P \in \mathcal{P}$ and all $Q \in \mathcal{P}_1$ simultaneously (among the polynomials $F(x)$ and x). Hence $G(x)$ is irreducible in $\mathrm{Int}(D)$ by Lemma 3.7.

Finally, again by Lemma 3.7, $H(x)$ is irreducible in $\mathrm{Int}(D)$, since

(i) $F(x)$ and $x - a_i$ are indispensable for P_i $(1 \leq i \leq n)$
(ii) $F(x)$ is indispensable for Q_i $(1 \leq i \leq m)$

among the polynomials $F(x)$ and $x - a_j$ with $1 \leq j \leq n$. □

As discussed at the beginning of this section, we may conclude:

Corollary 5.1. *Let D be a Dedekind domain with infinitely many maximal ideals, all of them of finite index.*

Then there does not exist a transfer homomorphism from the multiplicative monoid $\mathrm{Int}(D) \setminus \{0\}$ *to a block monoid; in other words:* $\mathrm{Int}(D)$ *is not a transfer Krull domain.*

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