

## 1. Extension-Contraction and the Spectrum of a Polynomial Ring

For subsets  $A, B$  of a ring  $R$ , we define

$$AB = \{a_1b_1 + \dots + a_nb_n \mid n \in \mathbb{N}, a_i \in A, b_i \in B\}.$$

For a subset  $S$  of a commutative ring  $R$  with 1, the ideal of  $R$  generated by the set  $S$  is  $SR$ .

\*\*\* *Extension and Contraction of Ideals* \*\*\*

**1.1 Definition.** If  $R \subseteq S$  are commutative rings and  $1_S = 1_R$ , we say  $S:R$  is a **ring extension**.

**1.2 Definition.** Let  $S:R$  be a ring extension. Then every ideal  $I$  of  $R$  defines an ideal of  $S$  by **expansion**:  $I^e$  is the ideal of  $S$  generated by  $I$ , that is,  $I^e = IS$ .

Every ideal  $J$  of  $S$  defines an ideal of  $R$  by **contraction**:  $J^c = J \cap R$ .

More generally, if  $f: R \rightarrow S$  is a ring-homomorphism, then every  $I \trianglelefteq R$  defines an ideal  $I^e$  of  $S$ : the ideal of  $S$  generated by  $f(I)$ , that is,  $I^e = f(I)S$ . Every ideal  $J$  of  $S$  defines an ideal of  $R$  via contraction:  $J^c = f^{-1}(J)$  of  $R$ . When  $f$  is understood,  $I^e$  is sometimes written as  $IS$  and  $J^c$  as  $J \cap R$  by abuse of notation even if  $f$  is not injective.

Contraction preserves the property of being a prime ideal (or a primary ideal); extension, in general, does not.

Clearly, for ideals  $A, B$  of  $R$  and  $C, D$  of  $S$ ,  $A \subseteq B \Rightarrow A^e \subseteq B^e$  and  $C \subseteq D \Rightarrow C^c \subseteq D^c$ ; and  $A^{ec} \supseteq A$  and  $D^{ce} \subseteq D$ .

Therefore, for every ring homomorphism  $f: R \rightarrow S$ , extension and contraction give a bijective correspondence between ideals of  $R$  of the form  $J^c = J \cap R$  ( $J$  an ideal of  $S$ ) and ideals of  $S$  of the form  $I^e = IS$  ( $I$  an ideal of  $R$ ). In two notable cases, all ideals of  $S$  are of the form  $I^e$ :

- Let  $H$  be an ideal of  $R$  and  $\pi: R \rightarrow R/H$ ,  $\pi(r) = r + H$ , the canonical projection. Then extension and contraction induce a bijection between all ideals of  $R/H$  and the ideals of  $R$  containing  $H$ . In both directions of this bijection the properties of being a prime ideal, a primary ideal, or a G-ideal are preserved. Note that for a subset  $S$  of  $R/H$   $\pi^{-1}(S) = \bigcup_{r+H \in S} r + H$ .
- Let  $S$  be a multiplicative subset of  $R$ ,  $R_S$  the ring of quotients with denominators in  $S$  and  $f: R \rightarrow R_S$  the canonical map  $f(r) = r/1$  (if  $R$  has a unit element, or  $f(r) = rs/s$  otherwise). Recall that  $f$  is injective whenever  $S$  contains no

zero-divisors. Then extension and contraction induce a bijection between all ideals of  $R_S$  and those ideals of  $R$  that are disjoint from  $S$ . Most importantly, if  $S = R \setminus P$ ,  $P$  a prime ideal, then the bijection is between all ideals of  $R_P$  and the ideals of  $R$  contained in  $P$ .

Recall that for  $I$  an ideal of  $R$ , the rings  $R[x]/I[x]$  and  $(R/I)[x]$  are canonically isomorphic via  $a_0 + a_1x + \dots + a_nx^n + I[x] \mapsto (a_0 + I) + (a_1 + I)x + \dots + (a_n + I)x^n$ .

\*\*\* *Spectrum of a Polynomial Ring* \*\*\*

Let  $R$  be a polynomial ring and  $P$  a prime ideal of  $R$ . Then every prime ideal  $Q$  of  $R[x]$  with  $Q \cap R \supseteq P$  contains  $P[x]$  and is therefore of the form  $\pi^{-1}(\bar{Q})$ , where  $\pi: R[x] \rightarrow R[x]/P[x] = (R/P)[x]$  is the canonical projection and  $\bar{Q}$  is a prime ideal of  $(R/P)[x]$ . Those  $Q$  among them with  $Q \cap R = P$  correspond to those  $\bar{Q}$  for which  $\bar{Q} \cap (R/P) = (0 + P)$ .

Let  $D$  be a domain with quotient field  $K$ . Then every prime ideal  $Q$  of  $D[x]$  with  $Q \cap D = (0)$  is of the form  $\bar{Q} \cap D[x]$ , where  $\bar{Q}$  is a prime ideal of  $K[x]$  (and thus  $\bar{Q} = (f)$ , where  $f$  is an irreducible polynomial of  $K[x]$ ).

To summarize: to determine for a given prime ideal  $P$  of  $R$  all prime ideals  $Q$  of  $R[x]$  with  $Q \cap R = P$ : consider the polynomial ring  $K_P[x]$ , where  $K_P$  is the quotient field of  $R/P$ , i.e.  $K_P = (R/P \setminus \{0 + P\})^{-1}R/P$  for each monic irreducible  $f \in K_P[x]$ , consider all polynomials of  $R/P[x]$  that are divisible by  $f$  in  $K_P[x]$ , i.e. all products  $fg$  with  $g \in K_P[x]$ , such that  $fg \in R/P[x]$ . Then the collection of all polynomials of  $D[x]$  whose residue class (under the projection of coefficients from  $R$  to  $R/P$ ) is of the form  $fg$  (for some  $g \in K_P[x]$ ) is a prime ideal of  $D[x]$  and all prime ideals  $Q$  of  $D[x]$  with  $Q \cap R = P$  are of this form.